Complex Analysis I

A. Describing Complex Numbers

$z = x + iy = re^{i\theta}$ Is a complex #. θ can run from - infinity to infinity. But each range θ+2π describes the same #.

$\prod_i z_i \prod_j z_j^{-1} = \prod_i r_i e^{i\theta_i} \prod_j r_j^{-1} e^{-i\theta_j}$ So we can see that products and divisions of complex numbers give complex numbers, and moreover the value doesn't depend on the particular branch (range of 2π) that θi,or θj resides in. So products of complex numbers are single valued.

The use of the phi + k2pi helps you see where the extra numbers come in but you should, if you can, really just think of it w/o the k2pi; instead just think of phi itself steadily increasing. The 2pi's will just get in the way when you multiply numbers together.

So we see that products and divisions of complex numbers are unique; therefore, functions that can be expressed as a taylor series are unique - like sin and cos ? Now consider complex numbers raised to fractional powers.

$z^{m/n} = r^{m/n} e^{mi\theta/n}$ In this case, a new value is obtained for each new revolution of θ, up to n-1 revolutions. So there are n branches of this function. Note that we would've obtained the same result using a Cartesian representation of the complex numbers, but it would've been less perspicuous.

We can extend this to any real power and now we'll see that there are infinitely many branches..

$z^c = r^c e^{ci\theta}$

And finally we'll consider a complex number raised to a complex number.

$z^i := e^{\ln\left(z^i\right)}$ $\quad e^{i\cdot\ln(z)}$ $\quad e^{i\cdot\ln\left[r\cdot e^{i\cdot(\phi)}\right]}$ $\quad e^{i\cdot\ln(r)-(\phi)}$ $\quad e^{i\cdot\ln(r)}\cdot e^{-(\phi)}$ you can see that z^i is unique.

$z^{a+b\cdot i} := z^a\cdot\left(z^i\right)^b$

Well it looks rather that it is not unique since there is not an i on the ϕ.

B. Complex Functions

B.1 Single Valued & Multivalued Functions

A single valued function which doesn't have any singularities other than poles in the entire plane is called a **meromorphic** function. Examples include rational polynomials, exp(z), sin(z), etc.

In the polar representation $0 \le r \le \infty \quad -\infty < \phi < \infty$

In the cartesian representation x, and y range over all the real numbers

This freedom is really the same as that for x and y and it often results in multivalued functions, which change values after ϕ sweeps around 2π, and really just relations. A branch cut is needed -- it restricts the values of ϕ to some range of 2π in the complex plane so that the function is single valued and analytic. When you pick a branch cut - a domain of 2pi, you agree to use it for every complex number, and product, power, function, etc. of complex numbers you have in your expression.

Functions can be expressed as f(z) or $f(x,y) := u(x,y) + i\cdot v(x,y)$

z^n is a single valued function for all ϕ

z^p is a multivalued function up until n revolutions (where p = m/n) when it starts to repeat again - so then you're in the same branch

z^c is multivalued function for all ϕ

$e^{i\cdot\phi} := \cos(\phi) + i\cdot\sin(\phi)$ is single valued

and so can see that it's single valued

$e^z := e^{x+i\cdot y} \quad e^x\cdot(\cos(y) + i\cdot\sin(y))$ is also single valued $\quad e^z := e^{r\cdot e^{i\cdot\phi}} \quad e^{r\cdot(\cos(\phi)+i\cdot\sin(\phi))} \quad e^{r\cdot\cos(\phi)}\cdot e^{i\cdot r\cdot\sin(\phi)}$

$a^z := e^{z\cdot\ln(a)} \quad e^{\ln(a)\cdot\left(r^{i\cdot\phi}\right)} \quad e^{\ln(a)\cdot r\cdot(\cos(\phi)+i\cdot\sin(\phi))} \quad e^{\ln(a)\cdot r\cdot\cos(\phi)}\cdot e^{i\cdot\ln(a)\cdot r\cdot\sin(\phi)}$ so this is also single-valued

I guess if you raise a number to a single valued function, then it must be single valued as well.

$$\cosh(z) := \frac{e^z + e^{-z}}{2} \quad \sinh(z) := \frac{e^z - e^{-z}}{2} \quad \tanh(z) := \frac{e^z - e^{-z}}{e^z + e^{-z}}$$ Are all single valued

$$\cos(z) \equiv \frac{e^{i\cdot z} + e^{-i\cdot z}}{2} \quad \sin(z) := \frac{e^{i\cdot z} - e^{-i\cdot z}}{2\cdot i}$$ Are single valued as well, and obey all the usual trigonometric identities

$$\tan(z) := \frac{\sin(z)}{\cos(z)}$$

(The sum, product, division of two single valued complex functions must be single valued itself)

And we have some more multivalued functions. ln(z) is multi-valued, and consequently so are the inverse trig and hyperbolic functions. We can solve for these in terms of the ln(z) function.

$$w = \sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \quad \Rightarrow \quad e^{2iz} - 2iwe^{iz} - 1 = 0$$

$$e^{iz} = \frac{2iw + \sqrt{-4w^2 + 4}}{2} = iw + \sqrt{1 - w^2}$$

$$z = \frac{1}{i}\ln\left(iw + \sqrt{1 - w^2}\right)$$

$$\sin^{-1}(z) := -i \cdot \ln\left(i \cdot z + \sqrt{1 - z^2}\right) \qquad \sinh^{-1}(z) := \ln\left(z + \sqrt{1 + z^2}\right)$$

$$\cos^{-1}(z) := -i \cdot \ln\left(z + \sqrt{z^2 - 1}\right) \qquad \cosh^{-1}(z) := \ln\left(z + \sqrt{z^2 - 1}\right)$$

$$\tan^{-1}(z) := \frac{1}{2i} \cdot \ln\left(\frac{1 + iz}{1 - iz}\right) \qquad \tanh^{-1}(z) := \frac{1}{2} \cdot \ln\left(\frac{1 + z}{1 - z}\right)$$

So we see that

$$\sinh^{-1}(iz) = i\sin^{-1}(z)$$
$$\cosh^{-1}(iz) = \cos^{-1}(z)$$
$$\tanh^{-1}(iz) = i\tan^{-1}(z)$$

B.2 Multi-valued functions and restricting them to single values

Consider the function

$$f(z) = \sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$$

and we see that the value of this function depends on the convention we use to specify the argument of z. So to keep the function single valued we need a method of describing all points in the Argand plane, and choosing one value of the function to assign to it. In this case we could say 0 < arg(z) < 2π, which I'll refer to as A.V. (arc value) or -π < arg(z) < π (P.V.), etc. Consider

$$\sqrt{z(z-1)}$$

To make this a single valued function we can use the ray method - treat z and z - 1 as vectors going from 0 to z and from 1 to z. Then we would possibly limit the range that the arguments of z and z - 1 may encompass. Another possibility is to simply commit to taking the P.V. z(z - 1), and take the square root then, etc. Either way we do this, you will see that there will be lines (branch cuts in space) which if we cross we will get a discontinous jump in the function.

$$\ln(z) = \ln(r) + i\theta$$

For this function we can do the same, restrict theta to some range:

$$\cos^{-1}(z) = \ln\left[z + \sqrt{(z-1)(z+1)}\right]$$

For this we could commit to taking the P.V. of the SQRT, or use the ray method, and then follow that by taking the P.V. of ln.

$$\ln(z + iz^2 - 2)\sqrt{z(z-1)}$$

And for this we could do a combination of things. We can take the principal value of the SQRT, and perhaps the principal value of the ln. Though I suppose that we could also take the AV of the ln, etc. We shouldn't be restricted to any particular consistent convention I don't think.

B.3 Branch Points and Branch Lines & how they relate to methods of single-valuing described above.

Let's go back to the original function, specified so as to make it single valued.

$$f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2} \qquad 0 < \arg(z) < 2\pi$$

In particular we see that the value of the function just above and just below the x axis are different.

$$f(x + i0^+) = \sqrt{xe^{i\tan^{-1}(0/x)}} = \sqrt{x}$$
$$f(x - i0^+) = \sqrt{xe^{i(2\pi - \tan^{-1}(0/x))}} = \sqrt{x}e^{i\pi} = -\sqrt{x}$$

In other words, starting from the positive x-axis, and circling the origin to get to the underside of the axis, we come across a discontinuity established by our method of making the function single valued. So the origin is called a branch point. And the positive x axis is called a branch line. If we cross the branch line then we enter another branch of the function - so to speak - because we would get a different value of the function at the same point in the complex plane. Note that there are two branches - after the second revolution we come back to the original value - the first branch. Note also that if we had used the convention of restricted the argument to the P.V., then the branch line would've been the negative x-axis. In other words - circling the point is the bad thing, not just approaching a particular axis.

Branch cuts are generally lines in the Argand plane through which passage is forbidden to prevent passing into another branch of the function, thereby making it a real function and also possibly analytic. Points on the branch cut are still part of the Argand plane so you can still calculate the function evaluated as such points - it is of course specified how you label such points - cf. ranges for theta at the top of the page. However, all the points on the branch cut are non analytic for that function, since its discontinuous there.

Note we also see branches emerge if we use a 'rule' convention for specifying z. So we see that to each sort of rule for specifying a unique value of the function, there is a unique branch line, or set of branch lines, marking off discontinuities (i.e. branches) in the function. And sometimes it is more useful to think of the branch line as originating from your method of single - valuing the function. Other times its more useful to think of the branch points, and cuts as primary, and methods of assigning values as secondary. Because branch points are 'real' entities. For instance, regardless of how use choose a branch cut, you can't integrate a function along a contour that completely circles a branch point without jumping into the next branch - or at least encountering that discontinuity.

If we want to use the next branch, then we'd have that $2\pi < \arg(z) < 4\pi$, (ext. of AV) or $\pi < \arg(z) < 3\pi$ (ext. of PV) or some such thing.

Consider again picking out the various branches of a multivalued function. How do you do so.

Next consider the following function. We use the ray diagram stuff to make it single valued first. And then discuss the branches.

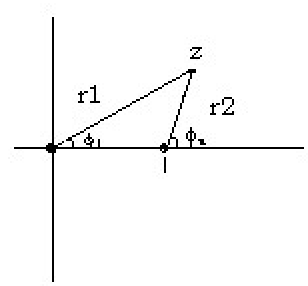
$$f(z) = \sqrt{z(z-1)}$$

Then referring to the diagram, for a given value of z in the complex plane: $z := r_1 \cdot e^{i \cdot \phi_1}$ $\quad z - 1 := r_2 \cdot e^{i \cdot \phi_2}$

Therefore $\sqrt{z}\sqrt{z-1} = \sqrt{r_1}\sqrt{r_2}e^{\frac{i}{2}(\phi_1+\phi_2)}$

We can start ϕ_1, ϕ_2 off at 0, or 2π on the positive real axis, according to which we prefer, but it has to be at multiples of 2π get the correct results. This is equivalent to setting the range of the angles to be 0 to 2π but then also including explicitly the generality $\phi 1 + 2\pi k$, $\phi 2 + 2\pi m$, and then specifying k + m. This is done in the contour integration section.

Notice that as z encircles O only, $\phi 1$ runs through a range of 2π and $\phi 2$ starts and ends at the same value, so it runs through a range of 0. Therefore an extra phase factor of π is picked up and w, passes to another branch. The same thing happens if you encircle z = 1. Here $\phi 1$ will run through a range of 0, and $\phi 2$ through a range of π. If you encircle both, then an extra 2π will be picked up, but that is eqivalent to 0, in the exponential so you get the same as before and therefore you are in the same branch.



Therefore a possible branch cut is a line segment connecting 0 and 1, though you could also do it with a ray going from 1 to ∞ and another from 0 to $-\infty$

Note that which ever branch cut you use, don't get the idea that somehow those rays drawn in the diagram can't intersect the branch cut, either at some point during along the path, or at the point z' itself. The only restriction is that starting from a point z, whose functional value is known, the angles you use in the above equation must be ones that are arrived at by making a path (that doesn't intersect the branch cut) from that point z to your other point. Otherwise the angles can be negative, positive, π, or $+,-100\pi$, as long as the **path** doesn't cross the branch cut. So for instance, as z (Re part > 1) crosses the real axis, $\phi 1$ will go from positive to negative, (not positive to 2π - something).
The rays are impervious to the branch cuts and the values of the angles they make will be the same as if the branch cuts weren't there. Only the point, z, can't cross the branch cut during its path.

Another way that you could specify the branch of this function is to commit to restricting the argument of z and z-1 to the range 0 to 2π. This is the same as above basically, as we saw with the first function. And another possibility is to multiply the two numbers together and then simply restrict the argument of the product to between 0 and 2π, i.e. the arc value. What branch cuts would this prescription correspond to? Let's see. Consider the function evaluated at 1+ i0 and 1 - i0.

$$f(2+i0^+)=\sqrt{(2+i0)(1+i0)}=\sqrt{2+3i0}=\sqrt{2}e^{i(\tan^{-1}0)/2}=2e^{i(0)}$$
$$f(2-i0^+)=\sqrt{(2-i0)(1-i0)}=\sqrt{2-3i0}=\sqrt{2}e^{i(2\pi-\tan^{-1}0)/2}=2e^{i\pi}$$

Clearly these are different. Now let's look at points inbetween.

$$f(1/2+i0^+)=\sqrt{(1/2+i0)(-1/2+i0)}=\sqrt{-1/4}=(1/2)e^{i(\pi)/2}=i/2$$
$$f(1/2-i0^+)=\sqrt{(1/2-i0)(-1/2-i0)}=\sqrt{-1/4}=(1/2)e^{i(\pi)/2}=i/2$$

These are the same. And now we'll look at points on the other side.

$$f(-2+i0^+)=\sqrt{(-2+i0)(-3+i0)}=\sqrt{6-5i0}=\sqrt{6}e^{i(2\pi-\tan^{-1}0)/2}=-\sqrt{6}$$
$$f(-2-i0^+)=\sqrt{(-2-i0)(-3-i0)}=\sqrt{6+5i0}=\sqrt{6}e^{i(\tan^{-1}0)/2}=\sqrt{6}$$

And they are different again. So it seems that our principle of multiplying the two together and then restricting that angle requires branch cuts extending from - infinity to 0 and from 1 to infinity. Note that if we took the principle value the branch cut would extend between 0 and 1 instead. So I guess we should use these lines as ones we cannot cross w/o making a jump discontinuity
in our function, which would render it non-analytic, and hence all the residue stuff we would like to apply invalid.

So should think like this, for practical purposes: Choose a unique method for evaluating the function. And then figure out where the resulting lines of discontinuity (branch cuts) are.

Next consider

$$\ln(z)=\ln(r)+i\theta$$

Note that we don't need to use the ray technique to see the multivaluedness of ln(z). If we merely specify that we always use $0 < \arg(z) < 2\pi$, we see that there is a disconuity above and below the positive x axis. Consider below and let x be real.

$$\ln(x+i0^+)=\ln\left[x\,e^{i\tan^{-1}(0^+/x)}\right]=\ln x$$
$$\ln(x-i0^+)=\ln\left[x\,e^{i\left(2\pi-\tan^{-1}(0^+/x)\right)}\right]=\ln x+2\pi i$$

So using a consistent method of specifying z evinces the need for the branch cut. Now consider acos(z). The typical way in which you would restrict these functions to single values is to agree to take a particular branch of the sqrt and then a particular branch of the ln. Let's do this and try to see where the branch points, and lines are.

Let's consider evaluating the function by taking arc value of (z-1)(z+1) and arc value of ln. And then let's see where branch cuts are. First let's consider the point z = +/- i0, suspecting that they originate from z = -1 and z = 1, either between them or towards infinity in some direction, let's try a point inbetween z = +/- i0.

$$
\begin{aligned}
\cos^{-1}(\pm i0^{+}) &= \ln(\pm i0^{+} + \sqrt{(\pm i0^{+} - 1)(\pm i0^{+} + 1)}) \\
&= \ln(\pm i0^{+} + \sqrt{(-1 \pm i0^{+})(1 \pm i0^{+})}) \\
&= \ln(\pm i0^{+} + \sqrt{-1}) \\
&= \ln(2 \pm i0^{+} + i) \\
&= \ln(2 + i \pm i0^{+}) \\
&= \text{same}
\end{aligned}
$$

So these are the same, and therefore there is no branch cut here. What about on the other side of z = 1. Let's try z = 2 +/- i0.

$$
\begin{aligned}
\cos^{-1}(2 \pm i0^{+}) &= \ln(2 \pm i0^{+} + \sqrt{(2 \pm i0^{+} - 1)(2 \pm i0^{+} + 1)}) \\
&= \ln(2 \pm i0^{+} + \sqrt{(1 \pm i0^{+})(3 \pm i0^{+})}) \\
&= \ln(2 \pm i0^{+} + \sqrt{3 \pm 4i0^{+}}) \\
&= \ln(2 \pm i0^{+} + \sqrt{3e^{i\begin{cases}4i0^{+}/3 \\ 2\pi\end{cases}}}) \\
&= \ln(2 \pm i0^{+} + \sqrt{3}e^{i\begin{cases}2i0^{+}/3 \\ \pi\end{cases}}) \\
&= \ln(2 \pm i0^{+} \pm \sqrt{3}) \\
&= \text{different}
\end{aligned}
$$

So there is discontinuity across the line going from z = 1 to infinity. What about z = -2 +/- i0?

$$
\begin{aligned}
\cos^{-1}(-2 \pm i0^{+}) &= \ln(-2 \pm i0^{+} + \sqrt{(-2 \pm i0^{+} - 1)(-2 \pm i0^{+} + 1)}) \\
&= \ln(-2 \pm i0^{+} + \sqrt{(-1 \pm i0^{+})(-3 \pm i0^{+})}) \\
&= \ln(2 \pm i0^{+} + \sqrt{3 \mp 4i0^{+}}) \\
&= \ln(2 \pm i0^{+} + \sqrt{3e^{i\begin{cases}2\pi \\ 4i0^{+}/3\end{cases}}}) \\
&= \ln(2 \pm i0^{+} + \sqrt{3}e^{i\begin{cases}\pi \\ 2i0^{+}/3\end{cases}}) \\
&= \ln(2 \pm i0^{+} \mp \sqrt{3}) \\
&= \text{different}
\end{aligned}
$$

Different again. So another branch cut would probably extend from -1 to -infinity.

Lets consider:

$$\tanh^{-1}(z) := \frac{1}{2}\cdot\ln\left(\frac{1+z}{1-z}\right)$$

This function can, I think, support two different branch cuts. One would be a line connecting the two branch points, and the other being lines going from the branch points to infinity. The first I believe will hold in this case because there is a ratio inside the ln and thus the arguments of the two functions will cancel as you go around the branch cut (the top will gain 2π, and so will the bottom). This wouldn't work for ln((1+x)(1-x)) because here the arguments would add, not cancel, to give you a nonzero result.

Now consider functions defined via an integral.

0. $$\int_{-\infty}^{\infty} dt \frac{1}{t^2+1}\frac{1}{t-z} \sim \frac{\pi}{z+i}$$

1. $$\int_a^b dt \frac{t}{\left(t^2+z^2\right)^{5/2}} \sim \frac{1}{\left(b^2+z^2\right)^{3/2}} - \frac{1}{\left(a^2+z^2\right)^{3/2}}$$

2. $$\int_a^b dt \frac{1}{t^2+z^2+1} = \frac{1}{\sqrt{z^2+1}}\left\{\tan^{-1}\left(\frac{b}{\sqrt{z^2+1}}\right) - \tan^{-1}\left(\frac{a}{\sqrt{z^2+1}}\right)\right\} = \frac{1}{\sqrt{z^2+1}}\frac{1}{2i}\ln\left[\frac{1+ib/\sqrt{z^2+1}}{1-ib/\sqrt{z^2+1}}\cdot\frac{1-ia/\sqrt{z^2+1}}{1+ia/\sqrt{z^2+1}}\right]$$

3. $$\int_0^z dt \frac{1}{\ln(t)} = \mathrm{li}(z)$$

4. $$\int_0^z dt \frac{1}{\sqrt{t^2+1}} \sim \sinh^{-1}(z) = \ln(z+\sqrt{z^2+1})$$

5. $$\int_0^z dt \frac{1}{t^2+1} = \tan^{-1}(z) = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$$

6. $$\int_0^z dt \frac{e^{-t}}{t}$$

Basically we can identify the branch cuts/branch points in the following manner. If z only in the integration limit, then we know df/dz and the branch points of f will be the simple poles or branch points of df/dz. If z is in the integrand though, then the branch points will occur for every z at which there is a branch point or simple pole in the integrand. Basically, you can solve for these points, z(t). Then given the domain of t (the region of integration), a range of z will be defined from z(t). This is where the branch cuts will lie I suppose. The end points of the range (z(a) and z(b)) are usually the branch points.

Keep in mind Morera's theorem that if a closed contour integral is zero, then the function is analytic inside the region.

(0) So z(t) = t and t ranges along the real line and so the branch cut is along the real line and branch points at +/- infinity

(1) We can see that t will encounter branch points at $z = \pm i\sqrt{|t|}$. And see so the branch cut ranges between $z = \pm i\sqrt{a}$, and $\pm i\sqrt{b}$

(2) Same issue as above - almost. Using Moreras theorem we would see that the integral over z would encounter poles at z = +/- i sqrt(t^2+1). And so the branch points will be at z = +/- i sqrt(a^2 + 1) and +/- i sqrt(b^2 + 1). Not sure how to get the ones at z = +/- i.

(3) We see that there will be branch points at the singularities of the integrand. Even if we didn't know how to integrate it. We should be able to see that if we integrate along a path that encircles the branch points of the integrand, we will pick up the same Perhaps one way to see this is to break the path integral into two parts. The first part would be an integral to a point close to the singularity. This is the well behaved part of the integral. And then from that point integrate around the singularity. This isolates the interesting part of the path integral which lies close to the desired singularity, and gives us a way of integrating the function near the singularity to see what behavior we should expect.

The integrand has a branch point at z = 0, and so we would infer that the integral has a branch point there naturally. Note however that the integrand is finite (0) there. Near z = 1, the integrand will be singular since ln(1) = 0. Close to this singularity, the integral will be of the form.

$$\frac{1}{\ln(z)} = \frac{1}{\ln(1-(1-z))} = \frac{1}{-(1-z)-(1-z)^2/2+\ldots}$$
$$= -\frac{1}{1-z}\left(\frac{1}{1+(1-z)/2+\ldots}\right) \approx -\frac{1}{1-z}$$

Thus the singularity is of the form 1/(1-t) and upon integration would turn into a ln term. Thus, I would expect that li(z) would also have a branch point at z = 1.

(4) Same as above. However this time, there are no branch points of the integrand, just singularities at z = + - i. Using the technique above, we can show that these will be branch points however, even if we didn't know how to integrate the whole thing.

4. $$\int_0^z dt \frac{1}{\sqrt{t^2+1}} \sim \sinh^{-1}(z) = \int_0^{i+\varepsilon} dt \frac{1}{\sqrt{(t+i)(t-i)}} + \int_{i+\varepsilon}^{\text{around i}} dt \frac{1}{\sqrt{(t+i)(t-i)}}$$
$$\approx \int_0^{i+\varepsilon} dt \frac{1}{\sqrt{(t+i)(t-i)}} + \int_{i+\varepsilon}^{\text{z around i}} dt \frac{1}{\sqrt{2i(t-i)}}$$

(5) We can again use the same technique as above, the form of the integrand near the singularities is inverse linear and therefore we expect ln's to come out, and consequently for these to be branch points.

(6) Ei(x) has a branch cut in the same fashion as lnx does

B.4 Specifying a Branch of a Multi valued function

$$f(z)=\sqrt{z(z-1)}$$

Suppose you make the above branch cut (between 0 and 1). Now you must determine which branch of the function you want to operate in. So you must specify which one of the possible values the function can assume at a point z, you want it to assume. This will determine what branch you're operating in. So for example, suppose you want the function to assume the real value 2^(1/2) at z = 2, then:

$$w(2) := \sqrt{2}\cdot e^{i\cdot\frac{1}{2}\cdot(\phi_1+\phi_2)} \qquad \text{must equal} \qquad \sqrt{2}$$

So we can set $\phi 1$ and $\phi 2$ equal to 0 on the positive real axis. Specifying a branch is the same as specifying a phase for the function. Specifying the particular value that a multivalued function takes, is also equivalent to this; so you don't exactly have to draw in your branch cut, etc. just specify which value the function should take at z, but this might run you into trouble when doing contour integration. If you specify a value of a particular branch of a function, then you have specified that branch, but if you want to determine the value of that function in the same branch, at other points, then the best way is, I think, to draw a ray at the point a, for each z-a in the function, determine the branch from those, and then make a path to the other point you want to calculate the function at.

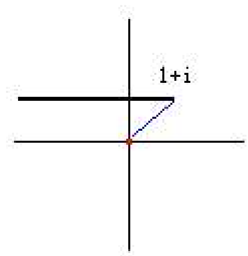
Another possible way is to simply agree to take the P.V. of the argument of any product of numbers under a root function, and/or similarly with the ln. For instance, with the function sqrt(z(z-1)). You can simply evaluate the product which is single valued, and then agree to take the argument of the product to lie within any certain range, 0 to 2π, $-\pi$ to π, or whatever, and then take the square root. Or you can demand that the phase of the number obtained have an argument between some different range. The choice you make determines the value of the function at a particular point.

This also makes me wonder what they mean exactly by specifying a branch by specifying the value of the function at a point. Since sqrt(x) will have the same value at 0 and different values at 1-i, in the two branches defined by restricting the argument to between 0 and 2π, and between $-\pi$ and π.

We generally choose the branch that makes a function real along a particular segment of the x-axis.

A.

$f(z)=\ln[z-(i+1)]$ branch cut is given by



Let us evaluate $f(0)$ This gives us

$$f(z)=\ln[-(i+1)]=\ln\sqrt{2}+i\arg[-(i+1)]$$

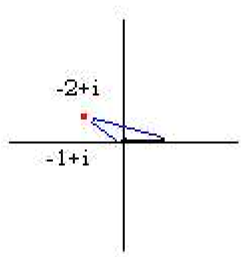
If we take the principal value, that is, assign the angles in the branch cut to run from $-\pi$ to π, we would have

$$f(0) = \ln\sqrt{2} - 3\pi i/4$$

We can also think of this as evaluating ln( ) as a function of z - (i + 1), so that z = 0 is indicated by the dot at the origin above. The argument of the ln is the blue line and so we evaluate

$$f(-1-i) = \ln\sqrt{2} + 3\pi i/4$$

B.



$f(z) = \sqrt{z(z-1)}$ branch cut is given by

Let us evaluate $f(-1+i)$ This gives us

$$f(-1+i) = \sqrt{(-1+i)(-2+i)}$$

Using the branch that we had discussed earlier - choosing the branch cut angles to be 0 along the positive real axis, this can be written

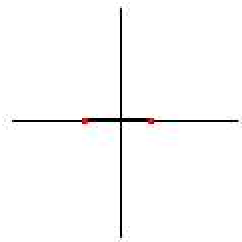
$$f(z) = \sqrt{z(z-1)} = \sqrt{r_1}\sqrt{r_2}e^{i\phi_1/2}e^{i\phi_2/2} = \sqrt{|-1-i|}e^{\frac{1}{2}i\arg(-1+i)}\sqrt{|-2+i|}e^{\frac{1}{2}i\arg(-2+i)}$$
$$= \sqrt{2}e^{\frac{1}{2}i3\pi/4}\sqrt{5}e^{\frac{1}{2}i\left[\pi-\tan^{-1}(1/2)\right]} = \sqrt{10}e^{\frac{1}{2}i\left[7\pi/4-\tan^{-1}(1/2)\right]}$$

Note that if we had decided to specify the branch by taking the *principal* (angles run from -π to π) value of the argument of the product, we would've come to a different answer

$$f(z) = \sqrt{z(z-1)} = \sqrt{1-3i} = \sqrt{10}e^{-i\tan^{-1}(3)}$$

C.

$$f(z) = \sin^{-1}(z) = -i\ln\left(zi + \sqrt{1-z^2}\right) = -i\ln\left(zi + \sqrt{-(z-1)(z+1)}\right)$$



We can use the branch cut to the left to specify the square root, then we can use a particular branch of the ln( ) function for the rest of the argument. I don't think there is any way to visually denote this 'extra' cut.

Let us use the branch of the square root such that when z = 0, the root = 1.

$$\sqrt{-(z-1)(z+1)} = e^{i\pi/2}\sqrt{1}e^{i\phi_1/2}\sqrt{1}e^{i\phi_2/2} = 1$$

Thus we can start ϕ1 off at 0 to 2π in the CCW direction, and ϕ2 from π to 3π in the CCW direction (which is a 2π shift from the usual case)

Here we have used i has the square root of (-1), instead of (-i). If I had used the other one, than I would use different ranges for the angles ϕ1 and ϕ2 below.

$$\sin^{-1}(1) = -i\ln\left(i+\sqrt{1-z^2}\right) = -i\ln(i) = 1$$

Upon taking the principle value of the argument of the ln( ) function.

B.5 Identities that hold for Real #'s now fail for Complex #'s, and looking for identities that do hold for every branch of a function

Consider the the manipulation

$$\frac{1}{i} = \frac{1}{\sqrt{-1}} = \sqrt{\frac{1}{-1}} = \sqrt{-1} = i$$

Certainly we can say SQRT(-1) = i if we wish, that's our choice. In that case though we're using a particular branch of the SQRT function. In that same branch it would be the case that SQRT(1) = 1 too. It's obvious that the problem lies with the middle step. Namely,

$$\sqrt{\frac{1}{-1}} \neq \frac{1}{\sqrt{-1}}$$

Now consider

$$\left.\begin{aligned} (z_1 z_2)^a &= r_1^a r_2^a e^{ia\left(\frac{(\theta_1+\theta_2)\bmod B}{2}\right)} \\ z_1^a z_2^a &= r_1^a r_2^a e^{ia\left(\frac{(\theta_1+\theta_2)}{2}\right)} \end{aligned}\right\} \rightarrow \left(z_1 z_2\right)^a \neq z_1^a z_2^a$$

where mod B means modulus the branch that that the number is supposed to be in. So we see that the first identity aren't equal since the sum, modulus of the branch doesn't have to be equal to the sum + 2πn. So powers don't distribute among their arguments. But note that if the numbers were real, then the identity would hold, as long as the numbers were being evaluated in the PB or AB. The first identity also wouldn't hold if z1 = z2 = z, regardless of whether it were evaluated in the PB or AB, etc. However, it could hold for special powers, certainly so if all powers are integers, or if the numbers are being divided, etc.

$$\left.\begin{aligned}(z^a)^b &= r^{ab}e^{ib[a\theta \bmod B]}\\ z^{ab} &= r^{ab}e^{iba\theta}\end{aligned}\right\}(z^a)^b \neq z^{ab}$$

This identity will not hold in general either. But an exception of b is an integer.

$$(z^a)^b = z^{ab} \qquad b \in \mathbb{Z}$$

Note that we can still use some identities in certain circumstances, or if we modify them. So for instance, if we keep everything in the principle branch, we may say,

$$\sqrt{\frac{1}{z}} = \sqrt{\frac{1}{re^{i\theta}}} = \sqrt{\frac{1}{r}e^{-i\theta}} = \frac{1}{\sqrt{r}}e^{-i\theta/2}$$
$$\frac{1}{\sqrt{z}} = \frac{1}{\sqrt{re^{i\theta}}} = \frac{1}{\sqrt{r}}e^{-i\theta/2}$$

So these two would be equal - except if $\theta = \pi$, in which case, $-\pi$ doesn't exist, and would go to π and so the top term would actually be $\exp(i\pi/2)$. And consider below, if we keep everything in the AB.

$$\sqrt{z \cdot z} = r\sqrt{e^{i(2\theta \bmod B)}} = re^{i\theta}\Theta(\pi-\theta) + re^{i\theta-\pi}\Theta(\theta-\pi) = \operatorname{sgn}(\pi-\theta)z$$
$$\sqrt{z}\sqrt{z} = re^{i\theta}$$

So these two aren't equal except under special circumstances. But consider

$$z^a z^b = r^{a+b}e^{i(a+b)\theta}$$
$$z^{a+b} = r^{a+b}e^{(a+b)\theta}$$

So exponents are additive regardless of the branch. So we may say,

$$\frac{t^{2/3}\sqrt{t}}{t^{5/2}} = t^{2/3+1/2-5/2} = t^{-4/3}$$
$$-1 = \sqrt{-1}\sqrt{-1}$$

Same goes for ln's now. If we evaluate ln via the P.V., A.V., or some other method, many identities won't hold. Since

$$\ln(z_1 z_2) = \ln r_1 + \ln r_2 + i(\theta_1 + \theta_2) \bmod B$$
$$\ln z_1 + \ln z_2 = \ln r_1 + \ln r_2 + i(\theta_1 + \theta_2)$$

$$\ln(z^n) = n \ln r + i(n\theta) \bmod B$$
$$n \ln z = n \ln r + i(n\theta)$$

$$\ln(z_1 / z_2) = \ln r_1 - \ln r_2 + i\left[(\theta_1 - \theta_2) \bmod B\right]$$
$$-\ln(z_2 / z_1) = \ln r_1 - \ln r_2 - i\left[(\theta_2 - \theta_1) \bmod B\right]$$

and these obviously won't be the same, though the one below will !

$$z = re^{i\theta}$$
$$e^{\ln z} = e^{\ln r + i\theta + 2\pi n} = e^{\ln r + i\theta} = re^{i\theta}$$

What we can say about these manipulations, is that each side is one of the possible values of the function. So the manipulations do map the set of possible values to the set of possible values in a 1-1 fashion. But they don't map the PV, AV, etc. of the LHS to the PV, etc. of the RHS. So they don't preserve branch I suppose you could say. So some identities will be preserved - per branch, and some won't. Though if we use the PV or AV of these functions with real # arguments, then they will hold.

So we can say that the following is true. These manipulations will preserve the branch.

$$z_1 = z_2 \rightarrow z_1^a = z_2^a$$
$$\rightarrow f(z_1) = f(z_2)$$

But we can't say, if f_inv(z) is a multivalued function, that

$$f^{-1}\left[f(z)\right] = z$$

We can only say that z is one of the many possible values that it can take on. Consider the solution of a quadratic equation now.

$$z^2 + pz + q = 0$$
$$(z + p/2)^2 + q - p^2/4 = 0$$
$$(z + p/2)^2 = p^2/4 - q$$
$$\sqrt{(z + p/2)^2} = \sqrt{p^2/4 - q}$$

So far everything is kosher, since the power 2 function conserves branch, and we can perform the same function on both sides as well.

$$z + p/2 = \sqrt{p^2/4 - q}$$
$$z = -p/2 + \sqrt{p^2/4 - q}$$

But this (top) move doesn't preserve branch. Since, as we found (z^a)^b doesn't equal z^ab. But let's say that we consider the some particular branch of the SQRT function. Then we can say that,

$$\sqrt{(z+p/2)^2} \in \pm(z+p/2)$$
$$\sqrt{p^2/4-q} \in \pm\sqrt{p^2/4-q}$$

And so

$$\pm(z+p/2) = \pm\sqrt{p^2/4-q}$$
$$z+p/2 = \pm\sqrt{p^2/4-q}$$
$$z = -p/2 \pm \sqrt{p^2/4-q}$$

So the upshot is that we can use the usual algebraic identities as long as we are careful to use the particular branch that makes the identity true, i.e., we may have to switch back and forth among the branches of the constituent terms to keep the overall values the same.

B.6 Multi - Valued Functions of Functions

Suppose that f(z) is multivalued and g(z) is single valued and we want to evaluate f(g(z)). Then we would evaluate z' = g(z) and then use whatever rules we have associated with f(z) to evaluate f(z'). Sometimes we can rewrite the rules for the function h(z) = f(g(z)) in terms of z instead of in terms of z' = g(z). For instance.

$$f(z) = \sqrt{z} + \sqrt{z-1}$$

has a branch cut along neg. infinity to 0 and from 1 to infinity. And suppose we take the arguments of z - 1 and z to be 0 to 2π and $-\pi$ to π respectively. Then,

$$f(-z) = \sqrt{-z} + \sqrt{-z-1} = \sqrt{-z} + \sqrt{-(z+1)}$$

And we could evaluate this by restricting -z and -z - 1 to their P.V. Or, keying on the fact that -1 maps the branch cut to between -infinity and -1 and 0 to infinity, we could do as follows. First agree to call -1, exp($i\pi$) or some such. That choice rotates everything CW by 180 degrees. So now z's argument ranges between 0 and 2π, and z-1 gets mapped to z+1 and its argument ranges between $-\pi$ and π, where $-\pi$ is on the underside of the branch cut, as we can see from,

$$0 < \arg(z-1) < 2\pi \to 0 < \arg(-z-1) < 2\pi \to 0 < \arg(e^{i\pi}(z+1)) < 2\pi$$
$$-\pi < \arg(z+1) < \pi$$

Or consider f(z) below withe the argument restricted to the P.V.

$$f(z) = \sqrt{z}$$

Then look at

$$f(-z)=\sqrt{-z}$$

-z would be restricted to the P.V., or, in other words. if we agree to call -1, exp(πi), then we'd have that z is restricted to the whole range (- 2π to 0). And note how multiplying by -1 would map the branch cut from the negative real axis to the positive real axis. And if we called -1, exp(-πi), then z would be restricted to 0 to 2π, etc. We could do the same thing with

$$f(iz)=\sqrt{iz}$$

And we'd find that the branch cut would be mapped from the negative real axis to the negative imaginary axis, and theta to between 3π/2 and -π/2, if we agree to call i, exp(iπ/2). So in general it rotates the coordinate system in the opposite direction by the specified angle. Note also that we don't need to say

$f(e^{i\pi/2}z)$ or $f(e^{5\pi i/2}z)$ whenever we mean $f(iz)$

For either of the three, or infinitely many possibilities are the same - they all are iz, since iz is a unique number. The question is, what do we do with the unique number iz to associate it to a unique f ? Do we restrict the argument to the P.V., or A.V., or a different branch altogether, like 3π < arg(iz) < 5π, etc.? So the point is that i doesn't have to be written in a certain way for us to make sense of f(iz). i can be written any way you want since iz = exp(iπ/2)z = exp(5πi/2)z = ...The question is, what does f require us to do with the number after we calculate it? So there is no ambiguity in saying f(iz).

Note that if we have a function like $\sqrt{z+iz^2-3}\cdot\sin^{-1}(\sqrt{z-1})$ then we would have to choose a branch of

$\sqrt{z+iz^2-3}$ and then choose a branch of $\sqrt{z-1}$ and then of $\sin^{-1}$

So we have many different independent options, and choosing a particular set will result in a different value of the function. But don't think that z is being described in different ways for each of the three functions. In fact, z is the same of all, but the value we associate with it depends on the branch we take. In this way, it might be better to think of z as just the #, and the function as the thing which assigns the particular description of the # (usually in terms of the argument). So for example SQRT(z) can have two values, not depending on the argument of z intrinsically, but on what argument the SQRT function assigns to z. In the first branch it assigns, for calculational purposes, the P.V., and in the second branch it goes from 0 to 2π. Also note that we might assign a branch to the 1st function, and one to the 2nd, and one to the 3rd, as a way of specifying a single value. We may not have individual branches for the entire function - just products of branches of the individual ones. Though this should be expressible, as before, as a single branch cut in the z - plane.

If we have a function $\ln(\sqrt{z^2+1})$ If we use the principle branch of the ln function, then we must have

$$\left|\arg(\sqrt{z^2+1})\right|<\pi$$

What does this look like?

B.7 Series Expansions

Consider an asymptotic expansion. We want to expand the function for large z, in the same branch.

$$\frac{\sqrt{z(1-z)}}{1+z^2}=\frac{\sqrt{-z^2(1-1/z)}}{1+z^2}=\frac{-zi\sqrt{1-1/z}}{z^2(1+1/z^2)}=-\frac{i}{z}\left(1-\frac{1}{2z}-\frac{1}{8z^2}-...\right)\left(1-\frac{1}{z^2}+\frac{1}{z^4}+...\right)$$

Note that in the third step I used sqrt(-1) = -i. This was necessary to stay in the same branch. For instance, if you plug 2 into the expession before any manipulations you get -sqrt(2)i. To get the same after the manipulations, you have to use sqrt(-1)=-i. The same sort of reasoning was necessary for the asymptotic expansion. There is a choice of phase in front of the expansion +/- 1. The +1 choice works for us. So, in general, after any step that doesn't preserve the branch automatically, you'll want to check that you made the right choices for sqrts and everything, so that you can be certain you're in the same branch.

Consider a Taylor series expansion near z = 1.

$$\frac{\sqrt{z(1-z)}}{1+z^2}=\frac{\sqrt{(z-1+1)(1-z)}}{1+z^2}=e^{i\theta}\frac{\sqrt{(z-1+1)}\sqrt{(1-z)}}{1+z^2}$$
$$=e^{i\theta'}\frac{\sqrt{1-z}}{1+z^2}\left(1+\frac{z-1}{2}+...\right)$$

We would expect the series to converge for |z-1| < 1, since we're expanding the function sqrt(z) about 1, and that function has a branch point at z = 0. Now for the choice of phase...Well, when 0 < z < 1, we would like our expansion to give a real #. So our choice of phase would be 0. So we'll have,

$$\frac{\sqrt{z(1-z)}}{1+z^2}=\frac{\sqrt{(1-z)}}{1+z^2}\left(1+\frac{z-1}{2}+...\right)$$

But can we actually assume that breaking the square root only costs us a phase? I think that we can be assured that this expression is accurate, at least because it gives the correct expression on the x axis between 0 and 1, and by analytic continuation, it must give the correct expression everywhere within its radius of convergence.

B.8 Integration & changing variables

Consider $\int_C dz\, z$

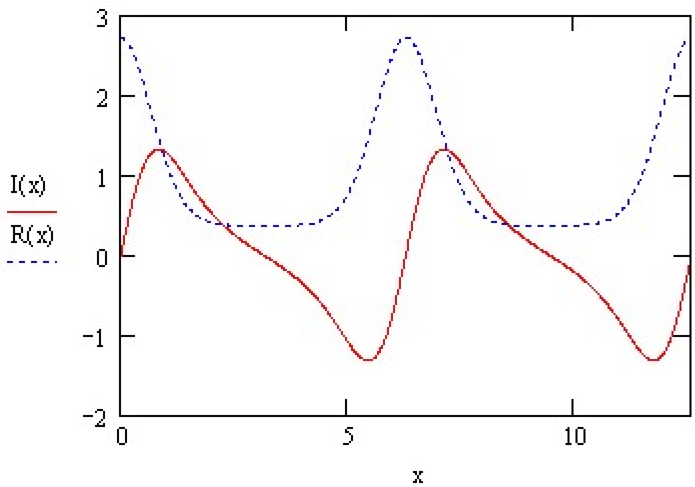
And suppose we change variables to z = ln(t) $\int_{C'} \frac{dt}{t}\ln t$

where we have a new contour C'. Suppose that the original contour C was given by

$C=z(t)=e^{i\theta} \quad \theta\in[0,4\pi]$ then the new contour is $C'=e^{z(t)}=e^{\cos\theta+i\sin\theta}=e^{\cos\theta}\left[\cos(\sin\theta)+i\sin(\sin\theta)\right]$

A map of the path is given below

$I(x) := e^{\cos(x)} \cdot \sin(\sin(x))$ $\qquad$ $R(x) := e^{\cos(x)} \cdot \cos(\sin(x))$



We note that there would be no problem in interpreting the path C' because that path doesn't encircle the origin ever - the real part is always positive. So we would simply use the P.V., or A.V. of our branch of the ln function and every little thing - is gonna be alright. But consider now

Consider $\displaystyle\int_C dz\, z$

And suppose we change variables to z = t^(1/3) $\displaystyle\int_{C'} dt \frac{t^{1/3}}{3t} t^{1/3} = \int_{C'} dt \frac{1}{3t^{1/3}}$

where we have a new contour C'. Suppose that the original contour C was given by

$C = z(t) = e^{i\theta} \quad \theta \in [0, 2\pi + \delta]$ then the new contour is $C' = z^3 = e^{3i\theta} \quad \theta \in [0, 2\pi + \delta]$

This new contour encircles the origin 3 times - going over each of the three branches. This is actually bad because the second integrand has a branch cut. So we'd have to choose some branch of the integrand and stay in it. Which one do we choose? The one that gives real z on the z-axis I suppose. So we could choose the P.V., or A.V. Consider the value of the second integral when one revolution is made using A.V.

$$\int_{1\,\text{rev.}} dt \frac{1}{3t^{1/3}} = \frac{1}{2}t^{2/3}\Big|_{t=1}^{t=1-i0} = \frac{1}{2}e^{4\pi/3} - \frac{1}{2}$$

$$\int_{3\,\text{rev.}} dt \frac{1}{3t^{1/3}} = 3\left(\frac{1}{2}e^{4\pi/3} - \frac{1}{2}\right)$$

Problem is, each revolution of the second integrand would give us this result and so at the end of three such revolutions we'd have a finite result, whereas in the first integral we'd have 0. Contrarily if we switched branches each revolution, we'd have

$$\int_{1\text{st rev.}} dt \frac{1}{3t^{1/3}} = \frac{1}{2}e^{4\pi/3} - \frac{1}{2}$$

$$\int_{2\text{nd rev.}} dt \frac{1}{3t^{1/3}} = \frac{1}{2}t^{2/3}\Big|_{t=e^{2\pi}}^{t=e^{4\pi}} = \frac{1}{2}e^{8\pi/3} - \frac{1}{2}e^{4\pi/3}$$

$$\int_{3\text{rd rev.}} dt \frac{1}{3t^{1/3}} = \frac{1}{2}t^{2/3}\Big|_{t=e^{4\pi}}^{t=e^{6\pi}} = \frac{1}{2}e^{12\pi/3} - \frac{1}{2}e^{8\pi/3}$$

$$\int_{3\,\text{rev.}} dt \frac{1}{3t^{1/3}} = \cancel{\frac{1}{2}e^{4\pi/3}} - \frac{1}{2} + \cancel{\frac{1}{2}e^{8\pi/3}} - \cancel{\frac{1}{2}e^{4\pi/3}} + \frac{1}{2}e^{12\pi/3} - \cancel{\frac{1}{2}e^{8\pi/3}} = 0$$

So that's an argument for leaving the branch. No matter what branch we choose, it would be the case that we'd simply be adding up identical finite integrals, for an integral in the t-plane which makes one revolution will never give 0, and thus arrive at a positive #. I suppose we could construct the rationale this way. When $\theta = 2\pi/3$, z = $\exp(2\pi\iota/3)$, t will be $\exp(2\pi i)$. And we expect that z and t^1/3 will give identical results - such will be the case so far. Next, when $\theta=4\pi/3$, z = $\exp(4\pi i/3)$ and t = $\exp(4\pi i) = \exp(2\pi i)$. And we want z = $\exp(4\pi i/3)$ to be equal to t^1/3 still. But this requires that we describe t as $\exp(4\pi i)$ and not as $\exp(2\pi i)$ as we would have if we stayed in the same branch. Finally, when $\theta=6\pi/3 = 2\pi$, z = $\exp(2\pi i)$ and t = $\exp(6\pi i) = \exp(2\pi i)$ again. But to get the correct result, we must again describe t as $\exp(6\pi i)$ not as $\exp(2\pi i)$. So in order to keep z and t^1/3 matched up, we're required to change branches of t^1/3 each time we make a revolution.

So generally speaking, we must be careful to match up values every time our changed variable crosses one of its branch cuts. It may be the case that the new integrand must changes branches there, or stay the same.

Let's look at the Ei(x).

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} dt \frac{e^t}{t}$$

There is a branch cut at perhaps the negative x axis, or positive - i.e. P.V., or A.V.

Now we consider Li(z)

$$\mathrm{li}(z) = \int_0^z dt \frac{1}{\ln t}$$

Now change variables to: t = exp(x)

$$\mathrm{li}(z) = \int_0^z dt \frac{1}{\ln t} = \int_{-\infty}^{\ln z} dx \frac{e^x}{x} = \mathrm{Ei}(\ln z)$$

Now to keep Ei well defined we have to use, say, its A.V., which means that we require 0 < arg(lnz) < 2π, if we're to use our result. This requires.

$$0 < \arg(\ln z) = |\arg(\ln r + i\theta)| < 2\pi$$

This will always be the case if lnr is positive and we use the P.V. of the ln function. There could be problems when ln(r) is negative. In any event, lets consider possible branch points. What do we get when we consider the difference between z = 1/2 +/- i0?

$$\mathrm{li}(0.5 \pm i0) = \mathrm{Ei}\left[\ln(0.5 \pm i0)\right] = \mathrm{Ei}\left[-\ln(2) \pm 0i\right]$$

So there would certainly seem to be no difference between the two points due to the branch cut in Ei. But consider the point 2 +/- i0.

$$\mathrm{li}(2 \pm i0) = \mathrm{Ei}\left[\ln(2 \pm i0)\right] = \mathrm{Ei}\left[\ln(2) \pm 0i \pm 2\pi ni\right] = \mathrm{Ei}\left[\ln(2) \pm 0i, 2\pi i\right]$$

Clearly a difference here

Now consider -x +/- 0i.

$$\mathrm{li}(-x \pm i0) = \mathrm{Ei}\left[\ln(-x \pm i0)\right] = \mathrm{Ei}\left[\ln(x) \pm \pi i\right]$$

And certainly there is a difference here as well. So we see the need for branch cuts between -infinity and 0 and between 1 and infiinity.

B.9 Differential Equations

Suppose we have the equation

$$\frac{\partial^2 y}{\partial z^2} + z\frac{\partial y}{\partial z} + (z^2 - 1)y = 0$$

And we want to change variables to:

$$z = \sqrt{1 + t^2}$$

We would need to specify that we're going to use the principal branch of the SQRT function in order to make this mapping 1-1.

Now let's solve for t,.

$$z = \sqrt{1+t^2} \rightarrow t = \pm\sqrt{1-z^2}$$

Where the SQRT is being evaluated in its principle branch. Now, which branch do we use the + or - sign? Well, just as when we have to choose whether to use t = +/-sqrt(1-z^2) when we make a change of variables, z = sqrt(1-t^2), we do here. In the latter case we use the fact that when t is positive, z should be too and vise versa. So we do too in the former case. So we choose the + sign.

$$t = \sqrt{1-z^2}$$

$$\frac{\partial}{\partial z} = \frac{\partial t}{\partial z}\frac{\partial}{\partial t}$$

$$\frac{\partial t}{\partial z} = 1/\frac{\partial z}{\partial t} = 1/\frac{t\sqrt{1+t^2}}{1+t^2} = \frac{1+t^2}{t\sqrt{1+t^2}} = \frac{\sqrt{1+t^2}}{t}$$ Recall that this identity holds for every branch

$$z^2 - 1 = \left(\sqrt{1+t^2}\right)^2 - 1 = \left(1+t^2\right) - 1 = t^2$$ Same here

etc.

C. Differentiation and Analyticity of Complex Functions

C.1 Definitions

Simple closed contour - one that separates Argand plane into one bounded region and one unbounded region - basically any closed loop that doesn't cross itself.

Connected Region - a region of points in Argand plane in which every point in the region can be connected to any other point in the region by a path that doesn't stray from the region. A Simply Connected Region is roughly a connected region with no holes. A Multiply Connected Region is one with holes.

A function is **analytic** at z0 if it differentiable there and throughout some neighborhood of that point.
It posesses a **singularity** at z0 if isn't differentiable there - this includes branch points - but is in at least one point in every neighborhood of z0

Isolated singularity - one that is analytic in every neighborhood of that point, excluding the point itself
Removable singularity - one that can be removed by a suitable definition of the value of the function there, for example sin(z)/z at 0. Note this is still a singularity, which has a residue, etc., even if you define f at that point?

C.2 Definition of the Derivative

Derivatives are defined the usual way $\quad f'(z0) := \lim_{z \to z0} \frac{f(z0 + \Delta z) - f(z0)}{\Delta z}$

Since the complex plane is 2-D, there are directional derivatives along the x and y axes (just take Δz to be Δx or Δy)
For the derivative to be well defined the two must be equal, that is the derivative is the same in all directions - no matter how you approach z0 - things must be relatively boring for the nice integral/differential properties of real numbers to carry over to the complex domain.

$$\frac{\partial}{\partial x}u + i\cdot\frac{\partial}{\partial x}v := -i\cdot\left(\frac{\partial}{\partial y}u + i\cdot\frac{\partial}{\partial y}v\right)$$

Which gives the Cauchy-Rieman conditions for Analyticity. This formula is easily visually derived by forming writing df, and then dividing by dz = dx, in the first case, and dz = idy in the second. Note that this basically requires that the derivative of f with respect to x is the same as its derivative with respect to iy, which will be the case provided that f can be written purely as a function of z = x+iy alone.

If u, v and their first partials with respect to x, and y are continuous in some neighborhood of z, then f is differentiable there iff the C-R equations are satisfied

$$\frac{d}{dz}f := \left(\frac{\partial}{\partial r}u + i\cdot\frac{\partial}{\partial r}v\right)\cdot e^{-i\cdot\phi} \qquad \frac{-i}{r}\cdot\left(\frac{\partial}{\partial\phi}u + i\cdot\frac{\partial}{\partial\phi}v\right)\cdot e^{-i\cdot\phi}$$

Expressed in polar coordinates   Where u and v are f(r,ϕ)

Note that this formally is easily visually derived just by imagining approaching z radially, then write df, and divide by dz, which in this case is just d(r*exp(iϕ)) in the radial direction which is exp(iϕ)*dr, and the first derivative formula is produced. For the second formula you approach z angularly. Write df, and then divide by dz = d(r*exp(iϕ)) = r*i*exp(iϕ)dϕ.

These conditions will be met if f can be written as a function of z = r*exp(iϕ) only.

LHospitals rule is valid in the complex domain, at least where both functions go to 0.
In addition the sum, product, quotient, and chain rules are valid. Moreover the derivatives of the common polynomial and trancedental functions carry over.

C.3 Differentiation of Multi - valued Functions

To take derivatives of multivalued functions recall the above discussion of expressing it as e^ln(f(z)); also you could, let w = f(z), solve for z in terms of w - which hopefully is a univalued function? and then take dz/dw. Next simply invert the derivative to get dw/dz, or df/dz and substitute in f(z) for w in the formula. These procedures will hopefully keep you in the same branch as the original function.

When you wish to evaluate the derivative of a multivalued function f(z), it is often convenient to express it as $f(z) := e^{\ln(f(z))}$

which is as we have seen, exactly true.

If $f(z) := (z-a)^p$ or something like that, then the branch used for f(z) is the same used for ln(f(z)), as you can see. So if you wish to evaluate f(z) in a particular branch, its easier perhaps to evaluate e(ln(f(z))) in that branch.

When you take derivatives of f(z), you want to stay in the same branch so it is essential that you write it out like the above. And that way you can be sure not to stray from the branch you're in - recall the case of sqrt(z1*z2) not equal to sqrt(z1)*sqrt(z2). That is, you can't just indiscriminantly add and subtract exponents when you have products and divisions of numbers, like you could in the real number case, and expect it to come out right, because those properties aren't true for multivalued functions in the same branch. And that is exactly what you would do if you took the derivative the old way. When differentiating functions make sure you stay within the same branch - which amounts to keeping your angles in the same range.Also if you want to evaluate Taylor series coefficients of multivalued functions - in a particular branch - you, of course, must do it this way. The same procedure holds true for any general f(z), and especially when you have f(z)^g(z). I think the branch of f(z) is the same as the branch of ln(f(z))? Think about it.

por ejemplo $\frac{d}{dz}(z-1)^{\frac{1}{2}}$ equals $\frac{d}{dz}e^{\ln(z-1)^{\frac{1}{2}}}$ equals $\frac{d}{dz}e^{\frac{1}{2}\cdot\ln(z-1)}$ equals $e^{\frac{1}{2}\cdot\ln(z-1)}\cdot\frac{\frac{1}{2}}{z-1}$

equals $\frac{1}{2}\cdot(z-1)^{\frac{1}{2}}\cdot\frac{1}{z-1}$

C.4 Harmonicity of Real and Imaginary Parts of Analytic function

Given an analytic function, f(z) in a simply connected domain D, its real and imaginary parts, u and v, are solutions to Laplace's equation in the complex plane. They are called harmonic functions. Conversely, if a function is harmonic on such a domain, then it is the real or imaginary part of a complex function that is analytic on that domain. The real and imaginary parts of that function form a mutually orthogonal family of curves.

D. Complex Integration

D.1 Integration Defined, and Fundamental Theorems

Integration along a contour C: $\int_C f(z)\,dz$ can be accomplished by parameterizing the contour $z(t) := x(t) + i\cdot y(t)$

for t between t0 and t1and calculating the integral $\int_{t_0}^{t_1} f(z(t(t)))\,dz$

So for instance, the following integral along a straight line contour would be

$$\int_0^{a+bi} dz\, z^2$$

$$x = at$$

$$y = bt$$

$$z = x + iy = at + ibt \rightarrow dz = (a+ib)dt$$

$$\int_0^1 (a+ib)dt\,(at+ibt)^2 = (a+ib)^3 \int_0^1 dt\, t^2 = \frac{(a+ib)^3}{3}$$

Or if we chose a contour that goes first to (x,y) = (a,0), and then to (x,y) = (a,b), we'd have

$$\int_0^{a+bi} dz\, z^2 = \int_0^{a+bi} (dx + idy)(x+iy)^2 = \int_0^a (dx)(x)^2 + \int_0^b (idy)(a+iy)^2$$

$$= \frac{a^3}{3} + i\frac{(a+iy)^3}{3i}\Bigg|_{y=0}^{y=b} = \frac{a^3}{3} + \frac{(a+ib)^3}{3} - \frac{a^3}{3} = \frac{(a+ib)^3}{3}$$

Given a simply connected, closed contour in the complex plane and an analytic function inside and on the contour, the line integral around the contour is always 0.

Deformation of Contours - very usefull for Laplace Transforms (because you can deform the integral you're told to do, to one where you just integrate around the singularities)

> It roughly states that you can continuously deform any line integral without changing its value as long you don't cross any singularities. That is, keeping the endpoints of the line integral glued down, you can contort the rest of the curve any way you want, as long as you don't cross a singularity. Also, consequently, a closed loop integral can be contorted anyway you want - can be continuously deformed - as long as you don't cross a singularity during the deformation. This is the idea behind the Wick rotation

Fundamental Theorem1

If in a domain that is simply connected, etc., and F(z) is analytic in it, then

$$\int_{z_0}^{z_1} f(z)\, dz := F(z_1) - F(z_0) \qquad \text{where} \qquad \frac{d}{dz}F(z) := f(z)$$

Note: for multivalued functions, in order to use the above theorem to calculate a line integral, you would have to make an appropriate branch cut that doesn't interfere with the path

Fundamental Theorem 2

If f(z) is analytic in some simply connected domain then

$$F(z) := \int_{z_0}^{z} f(z)\,dz$$ where the integral can be taken along any path in the domain

D.2 Important Theorems pertaining to Integrals

Cauchy Integral Formula: Let f(z) be analytic on and in a simple closed contour, and let z0 be in the interior of C, then

$$f(z_0) := \frac{1}{2\cdot\pi\cdot i}\cdot\int \frac{f(z)}{z - z_0}\,dz$$ where the integral is around the contour

Extension of Integral Formula: If f(z) is analytic inside C, then so are all its derivatives, which it posesses in all orders. Moreover

$$\frac{1}{n!}\cdot\frac{d^n}{dz^n}f(z_0) := \frac{1}{2\cdot\pi\cdot i}\cdot\int \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

again the integral is along the contour. Note that this result can be obtained by viewing it as a Residue problem. z0 is an n+1th order pole. Also note that you can derive this by differentiating with respect to z0. It can also be obtained from the formula above by differentiating n times.

Maximum/Minimum modulus Theorem: If f(z), a non constant function, is analytic inside a closed bounded region and continuous on it, the maximum value of its modulus must occur on the boundary. If the function doesn't equal zero anywhere inside the region, then the minimum will occur there also.

Liouville's thereom: An entire (anaytic everywhere), bounded f(z) must be constant.

Gauss's (Mean Value) Theorem : Let f(z) be analytic in a simply connected doemain, and z0 be inside the domain. Then f(z0) = the average of the values on a circle centered at z0. This follows from Cauchy's Integral formula.

$$f(a) = \frac{1}{2\pi i}\int_{circle} dz\,\frac{f(z)}{z-a} = \frac{1}{2\pi i}\int_{circle} rie^{i\varphi}d\varphi\,\frac{f(a+re^{i\varphi})}{re^{i\varphi}} = \frac{1}{2\pi}\int_{circle} d\varphi\, f(a+re^{i\varphi}) = \text{arithmetic mean of } f \text{ around circular contour}$$

D.2.1 Application of Gauss's Mean Value Theorem to Evaluation of Definite Integrals

A.

As an example of the usefulness of this theorem, consider the following integral.

$$\int_0^{2\pi} dx \ln(a + b\cos x)$$

We want to transform this integral into one of the form $\int_0^{2\pi} d\varphi f(c + de^{i\varphi})$

This is hard to do however, and we will take advantage of the following property of ln.

$$\begin{aligned}\int_0^{2\pi} d\varphi \ln(c + de^{i\varphi}) &= \int_0^{2\pi} d\varphi \ln\left|c + de^{i\varphi}\right| + \int_0^{2\pi} d\varphi \ \pi i \arg\left(c + de^{i\varphi}\right) \\ &= \int_0^{2\pi} d\varphi \ln\left|c + de^{i\varphi}\right| + 0 \\ &= 2\pi \ln(c)\end{aligned}$$

d is real, so the argument is that of points on a circle with radius d, centered at the point z = c on the real line. Thus you can see that the integral will be odd about phi = pi.

We can rewrite a + bcos(x) in the form of a modulus. So we can equate $a + b\cos\varphi = \left|c + de^{i\varphi}\right|^2 = c^2 + d^2 + 2cd\cos\varphi$

Doing this we come to

$$c^2 = \frac{1}{2}\left(a + \sqrt{a^2 - b^2}\right)$$

Then we note that

$$\begin{aligned}2\int_0^{2\pi} d\varphi \ln(c + de^{i\varphi}) &= 2\int_0^{2\pi} d\varphi \ln\left|c + de^{i\varphi}\right| + 2\int_0^{2\pi} d\varphi \ \pi i \arg\left(c + de^{i\varphi}\right) \\ &= 2\int_0^{2\pi} d\varphi \ln\left|c + de^{i\varphi}\right| + 0 \\ &= 2 \cdot 2\pi \ln(c) = 2\pi \ln(c^2)\end{aligned}$$

So therefore,

$$\int_0^{2\pi} d\varphi \ln\left|c + de^{i\varphi}\right|^2 = 2\pi \ln(c^2)$$

Therefore $$\int_0^{2\pi} dx \ln(a + b\cos x) = 2\pi \ln c^2 = 2\pi \ln\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right)$$

B.

As a last, and rather pertinent example, consider the integral

$$\int_{-b}^{b} dy \frac{\ln\left|x^2 - y^2\right|}{\sqrt{b^2 - y^2}}$$

We recognize that this integral doesn't depend on x at all, by differentiating with respect to x. Then we get

$$\int_{-b}^{b} dy \frac{2x}{(x^2 - y^2)\sqrt{b^2 - y^2}} = \int_{-b}^{b} dy \left\{ \frac{1}{(x-y)\sqrt{b^2 - y^2}} + \frac{1}{(x+y)\sqrt{b^2 - y^2}} \right\} = \int_{-b}^{b} dy \frac{2}{(x-y)\sqrt{b^2 - y^2}} = 0$$

which we already knew. So we can let x = 0 in full generality

$$\int_{-b}^{b} dy \frac{\ln\left|y^2\right|}{\sqrt{b^2 - y^2}}$$

To proceed, we make the standard change of variables to come to

$$\int_{-b}^{b} dy \frac{\ln\left|y^2\right|}{\sqrt{b^2 - y^2}} = \int_{\pi}^{0} d\varphi \ln\left|\cos^2 \varphi\right| = -\int_{0}^{\pi} d\varphi \ln\left|\cos^2 \varphi\right|$$

Now we make the change of variables to make the limits go to the standard 0 to 2π

$$-\frac{1}{2}\int_{0}^{2\pi} d\varphi \ln\left|\cos^2 \frac{u}{2}\right| = -\frac{1}{2}\int_{0}^{2\pi} d\varphi \ln\left|\frac{1}{2} + \frac{1}{2}\cos\varphi\right|$$

Comparing with our previous result, we have a = b = 1/2, which gives us

$$-\frac{1}{2}\int_{0}^{2\pi} d\varphi \ln\left|\frac{1}{2} + \frac{1}{2}\cos\varphi\right| = -\frac{1}{2} 2\pi \ln\left|\frac{1}{4}\right| = 2\pi \ln 2$$

D.3 Dispersion Relations

The real and imaginary parts of an analytic function are related by what are known as 'dispersion' or 'Kramers Kronig' relations. To develop these relations we consider first a function f(z) that has no singularities in the u.h.p. and also goes to 0 as abs.(z) goes to 0. Using Cauchy's integral formula for a semi-circular contour, C, we have.

$$\varphi(z) = \frac{1}{2\pi i}\int_{C} dz' \frac{\varphi(z')}{z' - z} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dy \frac{\varphi(y)}{y - z}$$

since the integral over the infinite semi - circle arc will go to zero as usual.

Now define our function of interest to be the value of $\phi(z)$ along the real axis - or, if there is a branch cut on the real axis - the limit of $\phi(z)$ as it approaches the real axis from above (or below I suppose). Note that this is still a complex function - just defined over real values. We have

$$\varphi(x) = \varphi(x+i\delta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dy\ \frac{\varphi(y)}{y-x-i\delta} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dy\ \frac{\varphi(y)}{y-x} + \frac{1}{2}\varphi(x)$$

$$\varphi(x) = \frac{1}{\pi i}\int_{-\infty}^{\infty} dy\ \frac{\varphi(y)}{y-x}$$

Now we equate the real and imaginary parts of this equation

$$\begin{aligned}\operatorname{Re}\varphi(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} dy\ \frac{\operatorname{Im}\varphi(y)}{y-x} \\ &= H\left[\operatorname{Im}\varphi(x)\right]\end{aligned} \qquad \begin{aligned}\operatorname{Im}\varphi(x) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} dy\ \frac{\operatorname{Im}\varphi(y)}{y-x} \\ &= -H\left[\operatorname{Re}\varphi(x)\right]\end{aligned}$$

Where H stands for the Hilbert transform. So if $\phi(z)$ is an analytic function in the U.H.P., or L.H.P. I suppose too, its real and imaginary parts on the x axis are related via these Hilbert transforms. Of course, given its analyticity, we may analytically continue the function from z on the real axis to z in the U.H.P./L.H.P.

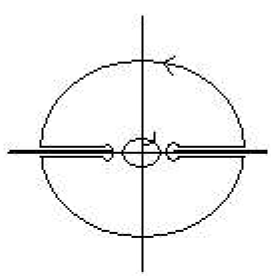
Now let us consider the use of these dispersion relations to represent functions with prescribed analytic and non - analytic behaviors.

Ex.1

Suppose that

1. f(z) is analytic everywhere except for a pole of residue 1 at z = 0, and has branch lines from 1 to infinity, and -1 to - infinity.
2. f(z) goes to 0 as mod(z) goes to infinity.
3. f(z) is real on the real axis from -1 to 1 (inbetween the brach cuts).

Then we apply the Cauchy formula along the contour shown below



$$\begin{aligned}\varphi(z) &= \frac{1}{2\pi i}\int_C dz'\,\frac{\varphi(z')}{z'-z} \\ &= \frac{1}{2\pi i}\oint_{|z|=\varepsilon} dz'\,\frac{\varphi(z')}{z'-z} + \frac{1}{2\pi i}\oint_{|z-1|=\varepsilon} dz'\,\frac{\varphi(z')}{z'-z} + \frac{1}{2\pi i}\oint_{|z+1|=\varepsilon} dz'\,\frac{\varphi(z')}{z'-z} + \frac{1}{2\pi i}\oint_{|z|=R} dz'\,\frac{\varphi(z')}{z'-z} \\ &+ \frac{1}{2\pi i}\int_1^{\infty} dy\,\frac{\varphi(y+i\delta)}{y+i\delta-z} - \frac{1}{2\pi i}\int_1^{\infty} dy\,\frac{\varphi(y-i\delta)}{y-i\delta-z} + \frac{1}{2\pi i}\int_{-\infty}^{-1} dy\,\frac{\varphi(y+i\delta)}{y+i\delta-z} - \frac{1}{2\pi i}\int_{-\infty}^{-1} dy\,\frac{\varphi(y-i\delta)}{y-i\delta-z}\end{aligned}$$

Now we will assume that the integrals at infinity and around the branch points go to zero.

$$\varphi(z)=\frac{1}{z}+\frac{1}{2\pi i}\int_{1}^{\infty}dy\frac{\varphi(y+i\delta)}{y+i\delta-z}-\frac{1}{2\pi i}\int_{1}^{\infty}dy\frac{\varphi(y-i\delta)}{y-i\delta-z}+\frac{1}{2\pi i}\int_{-\infty}^{-1}dy\frac{\varphi(y+i\delta)}{y+i\delta-z}-\frac{1}{2\pi i}\int_{-\infty}^{-1}dy\frac{\varphi(y-i\delta)}{y-i\delta-z}$$

Next, we use the Schwartz reflection principle that f(z*)=f(*(z) for analytic functions. This enables us to combine the the two integrals to find...

$$\begin{aligned}
\varphi(z)&=\frac{1}{z}+\frac{1}{2\pi i}\int_{1}^{\infty}dy\frac{\varphi(y+i\delta)}{y+i\delta-z}-\frac{1}{2\pi i}\int_{1}^{\infty}dy\frac{\varphi(y-i\delta)}{y-i\delta-z}+\frac{1}{2\pi i}\int_{-\infty}^{-1}dy\frac{\varphi(y+i\delta)}{y+i\delta-z}-\frac{1}{2\pi i}\int_{-\infty}^{-1}dy\frac{\varphi(y-i\delta)}{y-i\delta-z}\\
&=\frac{1}{z}+\frac{1}{2\pi i}\int_{1}^{\infty}dy\frac{\varphi(y+i\delta)}{y-z}-\frac{1}{2\pi i}\int_{1}^{\infty}dy\frac{\varphi(y-i\delta)}{y-z}+\frac{1}{2\pi i}\int_{-\infty}^{-1}dy\frac{\varphi(y+i\delta)}{y-z}-\frac{1}{2\pi i}\int_{-\infty}^{-1}dy\frac{\varphi(y-i\delta)}{y-z}\\
&=\frac{1}{z}+\frac{1}{2\pi i}\int_{1}^{\infty}dy\frac{\varphi(y+i\delta)-\varphi(y-i\delta)}{y-z}+\frac{1}{2\pi i}\int_{-\infty}^{-1}dy\frac{\varphi(y+i\delta)-\varphi(y-i\delta)}{y-z}\\
&=\frac{1}{z}+\frac{1}{2\pi i}\int_{1}^{\infty}dy\frac{2i\,\mathrm{Im}\,\varphi(y+i\delta)}{y-z}+\frac{1}{2\pi i}\int_{-\infty}^{-1}dy\frac{2i\,\mathrm{Im}\,\varphi(y+i\delta)}{y-z}\\
&=\frac{1}{z}+\frac{1}{\pi}\int_{1}^{\infty}dy\frac{\mathrm{Im}\,\varphi(y+i\delta)}{y-z}+\frac{1}{\pi}\int_{-\infty}^{-1}dy\frac{\mathrm{Im}\,\varphi(y+i\delta)}{y-z}\\
&=\frac{1}{z}+\frac{1}{\pi}\int_{1}^{\infty}dy\frac{\Phi(y)}{y-z}+\frac{1}{\pi}\int_{-\infty}^{-1}dy\frac{\Phi(y)}{y-z}
\end{aligned}$$

In the third line we see that what determines the function at all values of z is simply the residue at z = 0, and the discontinuity across the branch cuts - amazingly. In the fourth line we use the Schwarz reflection principle. In the last line we just call $\phi(x+i\delta)$, $\Phi(x)$. Thus we have completely prescribed the function in terms of its properties aforementioned.

Now suppose that what we want is actually a dispersion relation for F(x). We can determine this by letting z go to $x + i\delta$

$$\begin{aligned}
\varphi(z)&=\frac{1}{z}+\frac{1}{\pi}\int_{1}^{\infty}dy\frac{\mathrm{Im}\,\Phi(y)}{y-z}+\frac{1}{\pi}\int_{-\infty}^{-1}dy\frac{\mathrm{Im}\,\Phi(y)}{y-z}\\
\varphi(x+i\delta)&=\frac{1}{x+i\delta}+\frac{1}{\pi}\int_{1}^{\infty}dy\frac{\mathrm{Im}\,\Phi(y)}{y-x-i\delta}+\frac{1}{\pi}\int_{-\infty}^{-1}dy\frac{\mathrm{Im}\,\Phi(y)}{y-x-i\delta}\\
\Phi(x)&=\frac{1}{x}+\frac{1}{\pi}\int_{1}^{\infty}dy\frac{\mathrm{Im}\,\Phi(y)}{y-x}+\frac{1}{\pi}\int_{-\infty}^{-1}dy\frac{\mathrm{Im}\,\Phi(y)}{y-x}+i\,\mathrm{Im}\,\Phi(x)
\end{aligned}$$

Note that the single $\mathrm{Im}\Phi(x)$ comes from the fact that if x > 1, or x < -1, only the first or second integral respectively will pick up the delta function. And if -1 < x < 1, then we can include $\mathrm{Im}\Phi(x)$ at no cost because by assumption, in this region it is equal to zero. We can now go on to obtain the dispersion relation by taking the real part of both sides of the equation. And if we take the imaginary part of both sides, we just end up with a tautology.

$$\mathrm{Re}\,\Phi(x) = \frac{1}{x} + \frac{1}{\pi}\int_{1}^{\infty} dy \frac{\mathrm{Im}\,\Phi(y)}{y-x} + \frac{1}{\pi}\int_{-\infty}^{-1} dy \frac{\mathrm{Im}\,\Phi(y)}{y-x}$$

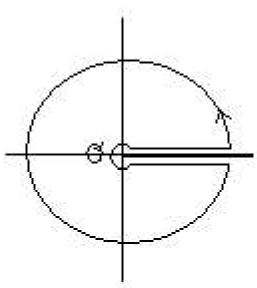
Note we could also have applied a 'semi circular' contour to obtain this relation instead the 'circular' one we used above.

Ex. 2

Find a function f(z) that has the following properties:

1. f(z) is analytic everywhere except for a branch line from z = 0 to infinity along the real axis and a simple pole of residue 1 at z = -1
2. f(z) goes to 0 as mod(z) goes to infinity.
3. f(z) is real on the negative real axis.
4. for x > 0, Imf(x+iδ) = 1/(1+x^2)

We use the following contour with Cauchy's theorem:



$$\begin{aligned}\varphi(z) &= \frac{1}{2\pi i}\oint_{|z+1|=\varepsilon} dz' \frac{\varphi(z')}{z'-z} + \frac{1}{2\pi i}\int_0^{\infty} dy \frac{\varphi(y+i\delta)}{y+i\delta-z} - \frac{1}{2\pi i}\int_0^{\infty} dy \frac{\varphi(y-i\delta)}{y-i\delta-z} \\ &= \frac{1}{z+1} + \frac{1}{\pi}\int_0^{\infty} dy \frac{\mathrm{Im}\,\varphi(y)}{y-z} = \frac{1}{z+1} + \frac{1}{\pi}\int_0^{\infty} dy \frac{1}{(1+y^2)(y-z)} \\ &= \frac{1}{z+1} - \frac{1}{\pi}\frac{\ln z}{(1+z)^2} - \frac{1}{1+z^2}\left(\frac{z}{2} - i\right)\end{aligned}$$

E. Complex Series

E.1 Definition of Series Expansion of a Complex Function

Let $S(z)_n := \sum_{i=1}^{n} u(z)_i$

$\lim_{n \to \infty} S(z)_n := S(z)$ for all z in some domain iff for any ε > 0, there exists a number N which can be N(z) such that |S(z) - S(z)n| < ε for all n > N(z). If there exists an N not a function of z, for all z in the domain then S(z)n is uniformly convergent; i.e. if the series converges at each point z.

$S(z)_n$ is absolutely convergent if $\sum_{i=1}^{n} \left|u(z)_i\right|$ is convergent, again N may be N(z), i.e. if at each point z you have an absolutely convergent series - def. here is extended from the usual real case

E. 2 Tests for convergence of series

$S(z)_n$ converges if both the real and imaginary parts converge. You can apply all the convergence tests for real numbers and functions by splitting S(z)n into its real and imaginary parts. You can do the same with the modulus, but keep in mind that S(z)n can converge, though the modulus doesn't

$S(z)_n$ is uniformly convergent if $\sum_{i=1}^{n} \left(\max \left|u(z)_i\right|\right)$ where max is over all z in domain -- converges

E. 3 Properties of Absolutely convergent series

1. Order of addition unimportant
2. Multiplication can be carried out as usual

E.4 Properties of Uniformly convergent series

1. $\sum_{i=1}^{n} \left(f(z)\cdot u(z)_i\right)$ converges uniformly if |f(z)| is bounded over the domain. Can interchange multiplication and summation.

2. If u(z)i is continuous over domain, then S(z) is too. $\sum_{i=1}^{n} \int u(z)_i\, dz$ converges uniformly to $\int S(z)\, dz$

   in domain. So can basically interchange the integration and summation

3. If u(z)i are analytic on domain, so is S(z), and the sum of the derivatives of u(z)i, converges uniformly to the derivative of S(z), and so you can basically interchange differention and summation

4. They are absolutely convergent

E.5 Definition of Power Series

If $u(z)_n := c_n\cdot\left(z - z_0\right)^n$ then S(z)n is a power series

E.5.1 Properties of Power Series

If S(z)n converges at z1, then it converges for all z such that |z - z1| is less than or equal to |z - z0|. In addition it is uniformly convergent and analytic in that domain.

If w(z) is analytic on the domain |z - z0| less than or equal to r, then there exists a power series uniformly convergent to w(z) on that domain expanded about z0.

Power series are unique

If w(z) is expanded in a power series about z0, and its nearest singlarity is z1, then the largest radius of convergence is |z1 - z0|

Equivalent Statements

$\frac{d}{dz}w(z)$ exists in some domain, D   w(z) has the only power series expansion valid in neighborhood of each point in D

$\frac{d^n}{dz^n}w(z)$ exists in D for all n

E.5.2 Techniques for obtaining power series

Of course the general Taylor series formula, familiar from real numbers holds here too. It can be derived just as in the Perturbation file. You can see that the coefficients are as given below, where C is a closed contour around z0, and of course no singularities if its a Taylor series. See the perturbation file.

$$a_n := \frac{1}{n!}\cdot\frac{d^n}{dt^n}f(z_0) \qquad a_n := \frac{1}{2\cdot\pi\cdot i}\cdot\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Another way to get the Taylor series formula when dealing with real numbers is to start with:

$$f(x) - f(x_0) := \int_{x_0}^{x} 1\cdot\left(\frac{d}{dt}f(t)\right)dt$$

and then succesively integrate by parts. Note that since you don't want to evaluate any of f's derivatives at t, but only t0, you cleverly give the antiderivative of 1 to be t - x, instead of just the usual t, and follow on to antidifferentiate it from there ... (t-x)^2/2, (t-x)^3/3, etc. Now I wonder if a similar procedure can here be used to determine the coefficients of the complex Taylor, or Laurent series'.

An easier way to develop the series is as follows:

Express w(z) as $f\left[(z-a)^n\right]$ where the Mclaurin expansion of f(z) is known and n > 0. Expand f(z) in a McLaurin series and substitute in (z-a)^n

Express the desired series as the integral, derivative, product, dividend, sum, etc. of another series - can do this b/c they're uniformly convergent.

Note that you divide complex series the same way you divide polynomials

If working with a rational function, decompose it into partial fractions and substitute in those simpler series

Make substitution w = x+a, and expand about 0

Note: $\frac{1}{1-z} := 1 + z + z^2 + z^3 + \ldots$ for all z such that $|z| < 1$

$\frac{1}{(1-z)^2} := 1 + 2\cdot z + 3\cdot z^2 + 4\cdot z^3 + \ldots$ for all z such that $|z| < 1$

Taylor Series of functions with branch cuts

Consider the function $\sqrt{z(1-z)}$

and consider that the particular branch of this function we are using gives real values between 0 and 1. We can expand this function in a Taylor series about, say, 0, as usual but we have to be careful that we stay in the same branch.

$$\sqrt{z(1-z)} = \sqrt{z}\sqrt{1-z} = \sqrt{z}\left(1-\frac{1}{2}z-\frac{1}{8}z^2-\ldots\right)$$

Note that the generally illegal second step is OK in this branch as you can see by plugging in z = 1/2 as a special case.

We can also expand this in a series about infinity.

$$\sqrt{z(1-z)} = \sqrt{-z^2(1-1/z)} = z\sqrt{-(1-1/z)} = iz\sqrt{1-1/z} = -iz\left(1-\frac{1}{2z}-\frac{1}{8z^2}-\ldots\right)$$

The second equality is valid in our branch. For the third equality I took as the square root of -1, i. By rewriting 1-1/z as (z-1)/z, you can see that this choice results in the correct value for z = 1/2. The expansion at the very end is valid for z > 1, so if we plug in z = 2, we can see that it also gives the same result as our original function in its particular branch. Note I had to put in a negative sign to keep the sqrt expansion in the same branch. Otherwise the two end results wouldn't give the same answers.

E.6 Laurent Series

If f(z) is analytic between two circular contours concentric around z0 - denote the region as R, then f(z) has a uniformly convergent Laurent expansion valid about z0 in R. Note this puts no requirements on f's analyticity at z0 - it may have a finite pole, and essential singularity (inf. order pole), or no pole at all there.

$$w(z) := \sum_{n=-\infty}^{\infty} \left[c_n \cdot (z - z_0)^n\right]$$

is uniformly convergent in an annular region between two singularities of w.
It is not confined to the annular region between the first two singularities.
Different Laurent series will be convergent in different ring regions.

The principle part of a Laurent series is the part with negative powers

c_{-1} is called the residue of the function w(z)

w(z) is said to have a pole of order N at isolated singularity z0, if c_{-N} is the first non-zero coefficient in the Laurent series expansion

If z0 is a singularity of order N, then $\lim_{z \to z_0} \left[(z - z_0)^m \cdot w(z)\right]$ will equal k if m = N, 0 if m > N, ∞ if m < N

Note for example: $\dfrac{1}{(z - z_0)^N}$ has an Nth order pole at z0

If N = ∞, then the singularity is called an essential singularity

E.6.1 Techniques for obtaining Laurent series

You can in principle use the expression analogous to a[n] with Cauchy integral formula for the coefficients c[n]; this isn't an efficient way to do develop the Laurent series, but it is useful for mathematical manipulations.

$$f(z) = \ldots + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots$$

We use Cauchy's theorem to solve for the nth term

$$c_n = \frac{1}{2\pi i} \int_C dz \frac{f(z)}{(z-a)^{n+1}}$$

This holds for positive or negative n - check it.

The more efficient techniques for determining the laurent series of a function are the same as those for determining the taylor series, except now you can let the n assume all integer values. Different laurent series will be valid in different locations however. You can use the Taylor series radius of convergence to determine where your Laurent series converges. Note if you want a series valid about a singular point - i.e. a Laurent series, then you'll need in your series a (z-a)^-n term, and similarly if you want one about any finite ring. If you want one from say b to infinity, then you will have to arrange it so that your series has no n > 0 powers - that is, if the function itself goes to 0, otherwise it will blow up. Generally speaking, the further out your ring of convergence is, the greater must by your powers of -n. And again you can just infer the region of convergence of whatever laurent series you come up with from the taylor series radius of convergence.

Other ways... If the Laurent series has an nth order pole at z0, then you can multiply f(z) by (z-z[0])^n, and then it will be finite at z0, and then it will have a Taylor series which you can determine as usual. Once you have the Taylor series for (z-z[0])^n*f(z) about z0, just divide the whole thing by (z-z[0])^n to get the Laurent series for f(z). Or just think of it as you have to multiply the whole thing by (z-z[0])^n, then take n-m derivatives to get the c[-m] term by itself (and have to divide by (n-m)!, and then take the limit z goes to z[0]. The same idea is behind finding the coefficients of the Taylor series.

so $c_{-m} := \frac{1}{(N-m)!} \cdot \frac{d^{N-m}}{dz^{N-m}}\left[\left(z-z_0\right)^N \cdot f(z)\right]$ where N is the order of the pole at z0

F. Residues

F.1 Definition of the Residue

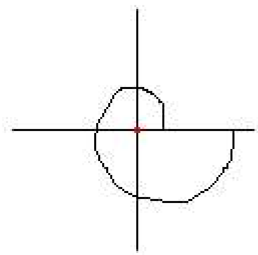
If w(z) is analytic on and in C except for at z0, then Res[w(z),z0] = $\frac{1}{2\cdot\pi\cdot i}\cdot\int w(z)\,dz$

where the integral is around z0, an isolated singularity, in the analytic domain. Note this excludes the cases when z0 happens to lie on a branch cut, or is a branch point. So the contour int. you usually do around them won't give you 2(pi)i*residue. z0 must be an isolated singularity.

Res[(wz),z0] = $\frac{1}{2\cdot\pi\cdot i}\cdot\sum_{i=-\infty}^{\infty}\left[c_i\cdot\int\left(z-z_0\right)^i dz\right]$ which is c_{-1} (notice) that the integrals will all vanish for i > -1, will equal $2\pi i$ for i = -1, and for the rest, will also equal zero as can be verified. Also note that the Laurent series constructed is convergent in that region.

Note this is a very general statement - whenever you integrate $\left(z-z_0\right)^n$ in a closed loop about z0, you always get 0 - regardless of course of whatever singularities of a function there are inside or outside the contour - except when n = -1, in which case you get 2pi*i. This is the key to evaluating integrals. Suppose you which to evaluate the integral of f around a contour which encompasses one or more of its singularities - if it doesn't encompass any then its 0. If the contour can be enclosed by an annulus in which f is analytic, then just expand f in a Laurent series about z[0], the center of the annulai. The only term in the series that will have a non zero integral is the (z-z[0])^-1 term, its integral will be 2pi*i. If any function can be expanded in a Laurent series valid between two annuli (which implies f(z) must be analytic between the two annuli, and remember that singularities of f inside or outside the region don't matter - all that is required for a convergent Laurent series is that f be analytic in the region between the annuli), then any closed integral in that region will just be 2pi*i*c[-1].

Let us note the following fact about residues, and the deformation of contours. Consider the following contour



I beg you to use your imagination here. But anyway, suppose we parameterize this (supposedly a nice smooth spiral) contour via

$$\varphi = t,\ r = 1 + t, \quad t = 0...2\pi$$
$$dz = d(re^{i\varphi}) = dte^{it} + (1+t)idte^{it}$$

and consider an integrand 1/z around this contour. The radius goes from 1 to 2π, and the angular variable ranges from 0 to 2π. The result will be

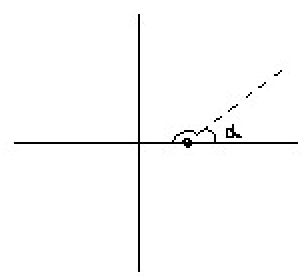
$$\int_C dz \frac{1}{z} = \int_0^{2\pi} \frac{dte^{it} + (1+t)idte^{it}}{(1+t)e^{it}} = \int_0^{2\pi} \frac{dt + (1+t)idt}{(1+t)} = \int_0^{2\pi} dt \frac{1}{1+t} + i = \ln(1+2\pi) + 2\pi i$$

And this is equivalent to integrating along a circular contour around the singularity at z = 0, with radius equal to 1, and then connecting this circular contour to z = 2π. The circular contour is the one responsible for the 2πi value, and the line integral from z=1 to z=2π would give you the ln(1+2π). This is easy to see by the deformation of contours theorem.

Fractional Residue about a point

In the limit as the radius of separation of the semi - circular arc contour around
the singularity (which must be a simple pole) goes to zero, the value of integral along the contour goes to

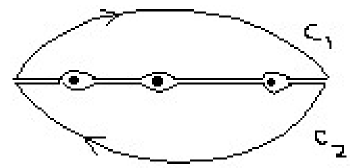
$$\frac{\beta}{2\cdot\pi}\cdot 2\cdot\pi\cdot i\cdot \mathrm{Res}\left(w(z), z_0\right)$$ where $\beta = 2\pi - \alpha$ and the direction of integration is assumed to be counter - clockwise



F.2 Use of the Residues in Evaluating Path Integrals

But its generally easier to evaluate a contour integral this way. If w(z) has a finite number of isolated singularities inside the closed contour C, but otherwise is analytic on and inside C, then

$$\sum_n \mathrm{Res}\left(w(z), z_n\right) := \frac{1}{2\cdot\pi\cdot i}\cdot\int w(z)\,dz$$ where integral is along C counter clockwise

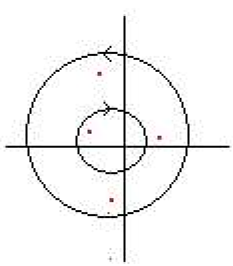


as can be seen by the diagram: As the separation goes to 0,

The integral along C = integral along C1 + integral along C2 + sum of integrals around the residues (notice that the integrals along the lines connecting the residues is zero since they are integrated twice in opposite directions and + is due to the residue integrals being in the opposite direction as the integral over C). Therefore after dividing everything by $2\pi i$, you get the above result

So sort of what we're doing is just expanding w(z) about each of its isolated singularities insided the contour in a Laurent series, and picking off the c-1 terms in each, adding them up, and multiplying by 2pi*i to get the integral of w(z) around the contour.

Concentric Contours



When we have two contours as shown, then the sum of the two line integrals is equal to $2\pi i$ times the sum of the residues inbetween the contours - as the outer contour is the sum of the residues inside it, and the inner integral is minus the sum of the residues inside it, since the inner contour goes in a clockwise direction.

F. 3 Techiques for Finding the Residue of a Function about a point

If w(z) has a pole of order 1 at z0, then

$$w(z) := c_{-1}\cdot\left(z-z_0\right)^{-1} + c_0 + c_1\cdot\left(z-z_0\right) + \ldots. \qquad \psi(z) := \left(z-z_0\right)\cdot w(z) \quad \text{equals} \quad c_{-1} + c_0\cdot\left(z-z_0\right) + c_1\cdot\left(z-z_0\right)^2 + \ldots.$$

$$\text{So}\quad \mathrm{Rez}\left(w(z), z_0\right) := \lim_{z \to z_0} \psi(z)$$

Moreover, if ψ can be expressed as f/g where g has a first order zero at z0 and f(z0) doesn't equal 0, then

$$\mathrm{Res}\left(\frac{f}{g}, z_0\right) := \frac{f(z_0)}{g(z_0)'}$$

If w(z) has a pole of order 2 at z0, then

$$w(z) := c_{-2}\cdot(z - z_0)^{-2} + c_{-1}\cdot(z - z_0)^{-1} + c_0 + c_1\cdot(z - z_0) + ...$$

$$\psi(z) := (z - z_0)^2\cdot w(z) \qquad \text{equals} \qquad c_{-2} + c_{-1}\cdot(z - z_0)^1 + c_0\cdot(z - z_0)^2 + c_1\cdot(z - z_0)^3 + ...$$

So $\mathrm{Rez}(w(z), z_0) := \lim_{z \to z_0} \frac{d}{dz}\psi(z)$

Moreover, if ψ can be expressed as f/g where g has a second zero pole at z0 and f(z0) doesn't equal 0, then

$$\mathrm{Res}\left[f(z)/g(z), z = z_0\right] = 2f'(z_0)/g''(z_0)$$

In general, if w(z) has a singularity of order N at z0, then

$$\mathrm{Rez}(w(z), z_0) := \lim_{z \to z_0} \left[\frac{1}{(N-1)!}\cdot\frac{d^{n-1}}{dz^{n-1}}\psi(z)\right]$$

This technique is generally applicable to obtaining every term in the Laurent series, if that is what you wish, assuming you start with an nth order pole.

If f(z) has singularity of order a at z0, and g(z) has one of order b there, then f(z)/g(z) will have one of order b - a there

F.4 Residues at infinity

The analytic behavior of w(z) at inf. is determined by the analytic behavior of w(1/z) at 0. w(z) is said to have a singularity at ∞ if w(1/z) has one at 0. Moreover, the order of the pole that w(1/z) posseses at 0 is considered to be the same as w(z) has at ∞. Also if w(1/z) is analytic at 0, w(z) is considered to be analytic at ∞. There is however, no connection between the concept of analyticity at infinity and the existence of residues at infinity, unlike in the finite case. This is because the residue of a function about a point is primarily the c[-1] coefficient in the Laurent series of that function valid about that point. While in the finite point case, to have a c[-1] coefficient, there must be a singularity there, in the infinite case, for example, 1/x has no pole there and yet has c[-1] = 1. So in general note that w(z) can be analytic at ∞ and yet have a residue there. According to the previous definitions, 1/x is completely analytic - no singularity whatsoever at infinity. This is conventionally expected because it and all of its derivatives exist at infinity. We'll also see that functions with singularities at infinity can also have residue's there.

F.4.1 Definition of Residue at infinity

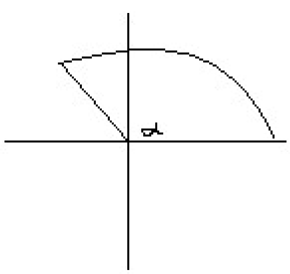
$$\mathrm{Res}(w(z),\infty) := \frac{1}{2\cdot\pi\cdot i}\cdot\int_C w(z)\,dz$$

Where C encloses all singularities of w(z) (not including the one at ∞). w(z) is considered to be analytic outside C, except at ∞. The integration is carried out along C in counterclockwise direction with respect to ∞, or clockwise with respect to a finite point. Note that if our contour integration is outside all finite singularities of the function, then this residue is the entire integral. Another reason for the clockwise definition is that the contour integral, evaluated clockwise will equal 2pi*i * sum of all residues outside the contour.

$\mathrm{Res}(w(z),\infty) := -c_{-1}$ where c_{-1} is the coefficient of the laurent series of w(z) valid in the region |z| > r, where r is large enough to include all the finite singularities in the region |z| < r. This laurent series is valid in a deleted neighborhood at ∞, analogous to series being convergent in a deleted neighborhood of a finite point. The fact that the integral equals -c_-1 is verified, just as it was for the finite singularity case. The only requirement was that the Laurent series be convergent to the function in that ring like region - whenever you expand a function about z0 into a Laurent series (in powers of (z-z0)) valid between two contours (surrounding z0), only the c-1 coefficient will contribute to the closed loop integration regardless of the presence of singularities or whatever inside or outside the contours. We get the minus sign here because the direction of integration is opposite.

So residues are defined by integrating around a 'singularity' CCW with respect to the singularity (and dividing by $2\pi i$). That is why the residue is c[-1] about finite singularity and -c[-1] about the infinite singularity.

Fractional Residue at infinity



If w(z) is analytic at ∞ with w(∞) = 0, and C is the arc of a sector (C is entirely on a circle) traversed clockwise, with angle α, and radius R > than distance to farthest finite singularity, then

$$\lim_{R\to\infty}\int_{\text{sector}} w(z)\,dz := \frac{\alpha}{2\cdot\pi}\cdot 2\cdot\pi\cdot i\cdot\mathrm{Res}(w(z),\infty)$$

similar to the theorem for finite residues, except analyticity part

F.4.2 Usefulness of the Residue at infinity for evaluating integrals

Theorems

If w(z) is analytic throughout the complex plane except possibly at a finite number of isolated singularities (including the one at infinity, if there is one), the sum of the residues of w(z) at the finite singularities + residue at ∞ will be zero.

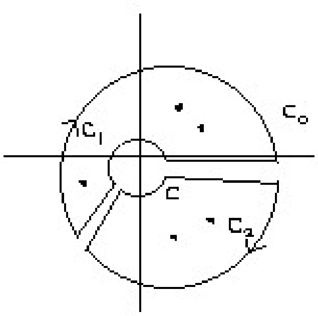
Residue of rational function whose denominator has a degree greater than the numerator by 2 or greater, has a residue at ∞ of zero. This is sensible because we find that the contour at infinity for rational polynomials goes to zero when the degree of the denominator is greater by 2 or more. And that semicircular contour at infinity is nothing more than 1/2 the residue at infinity. So this theorem is consistent with that fact.

Let w(z) be analytic on a simple closed contour C, and outside of C, except possibly at a finite number of isolated singularities (including the one at inf.). Note that there can be branch cuts inside C. Then

$$\int_C w(z)\,dz := 2\cdot\pi\cdot i\cdot\sum_n \mathrm{Res}\left(w(z), z_n\right)$$

where the residues are those outside of the contour, and C is clockwise.

Proof:



Let C be the contour spoken of above and C0 another contour which envelops all the singularities at finite points in the plane. Then cut the two contours at two places and create the two separate contours whose separation should be infinitely small, seen on the diagram.

$$\int_{C_1} w(z)\,dz + \int_{C_2} w(z)\,dz := -2\cdot\pi\cdot i\cdot\sum_n \mathrm{Res}\left(w(z), z_n\right)$$

$$\int_{C_1} w(z)\,dz + \int_{C_2} w(z)\,dz := \int_C w(z)\,dz + \int_{C_0} w(z)\,dz$$

$$\int_C w(z)\,dz + \int_{C_0} w(z)\,dz := \int_C w(z)\,dz + \mathrm{Res}(w(z), \infty)$$

$$\int_C w(z)\,dz + \mathrm{Res}(w(z), \infty) := -2\cdot\pi\cdot i\cdot\sum_n \mathrm{Res}\left(w(z), z_n\right)$$

which implies

$$\int_C w(z)\,dz := -2\cdot\pi\cdot i\cdot\sum_n \mathrm{Res}\left(w(z), z_n\right)$$

Including the one at ∞, and C is a CCW contour

So the integral in an analytic domain, inside, outside and on a closed contour (except for at the positions of the singularities in the finite plane) equals 2πi times the sum of the residues inside and -2πi times the sum of the residues (including at ∞) outside. When C is counter clockwise. So everything works out naturally; given a contour - its value equals 2πi times the sum of the inside residues if you go around it CCW with respect to those singularities. It also equals 2πi times the sum of the outside residues if you go around it CCW with respect to the outside singularites (or CW with respect to the inside singularites). The residue of any singularity is just the CCW integral around that singularity divided by 2πi - so you just find a laurent series valid about the the singularity and evaluate it CCW with respect to it. Note that it is this which makes the residue at inf. negative compared to the usual procedure - because CCW with respect to inf. is CW with respect to the O.

F.4.3 Techniques for Evaluating the Residue at infinity

One way is to expand the function, w(z), in a Laurent series valid in the region beyond all finite singularites, and pick off the c-1 coefficient. Other ways are..

If w(z) is analytic at ∞, or has a removable singularity there, and if $\lim_{z \to \infty} f(z) := 0$ then $\text{Res}(w(z), \infty) := \lim_{z \to \infty} \psi(z)$

where $\psi(z) = zw(z)$

We can also figure out f's residue at inf. by looking at its 'conjugate' function f(1/z). Note that this procedure is really only necessary when the laurent series of f(z) valid in the annulus beyond its last finite residue has an essential singularity at z = 0 (and consequently this would also be usefull for finding residues of functions with essential singularities at some finite point), or when, perhaps, the laurent series valid in that region is difficult to find. And this procedure is really only usefull probably when the laurent series - whatever it is - has a maximum positive z power. So just pretend that there is a maximum positive power for the following demonstration. in that case, you want to create a Laurent series about 0, like we're used to dealing with - with the c[-1] coefficient on the z^-1 power.

Consider $w(z) := ..... + c_{-2}\cdot z^{-2} + c_{-1}\cdot z^{-1} + c_0 + c_1\cdot z + c_2\cdot z^2 + ...$ valid for |z| > r

$$w\left(\frac{1}{z}\right) := .... + c_1\cdot z^{-1} + c_0 + c_{-1}\cdot z^1 + c_{-2}\cdot z^2 + ...$$ valid for |z| < 1/r

$$\frac{1}{z^2}\cdot w\left(\frac{1}{z}\right) := .... + c_1\cdot z^{-3} + c_0 z^{-2} + c_{-1}\cdot z^{-1} + c_{-2} + ...$$

Note that having an infinite pole changes the procedure a bit.

Notice that now you've got a Laurent series valid around 0 where the coefficient of 1/z is the residue desired. So all the previous tricks apply, just create f(1/z)/z^2 and find the residue at z = 0.

$$\mathrm{Res}(w(z),\infty) := -\mathrm{Res}\left(\frac{1}{z^2}\cdot w\left(\frac{1}{z}\right),0\right)$$

note that their are no requirements as to the convergence of f to 0 as z goes to ∞.