Complex Analysis II

G. Using Residues to evaluate definite integrals

G.1 Principal Value Integrals

The semi-circular contours that we make around singularities is for the purpose of evaluating the principal value of the integral. That is why we must make these contours around singularities that lie even on branch cuts. When you evaluate the integral by carrying out the integration on a line parallel to the real axis but shifted upwards by a small imaginary amount, you are calculating a different quantity than the principal value. We generally use this limit when doing Green's functions, etc. Also note that not all singularities are un-integrable.

$\ln(x), \dfrac{1}{x^{1-\alpha}} (\alpha < 1)$ are integrable singularities at 0.

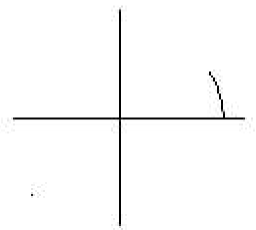
Not all singularities are integrable in the principal value sense though - $\dfrac{1}{x^2}$ is not integrable in that sense because both sides go to positive infinity. 1/x is integrable at 0 because the positive and negative infinities cancel each other out.

Note that the principal value of the integral $\displaystyle\int_a^b dt \frac{f(t)}{t-c}$

where c lies inbetween a and b, and f(c) is finite, doesn't pick up any finite amount from the singularity at c. In other words, you can just evaluate the antiderivative at b and subtract from the antiderivative evaluated at a, as usual. You can prove this by expanding f(t) in a Taylor series about t = c.

G.2 Some results on the asyptotic arc integrals of trig functions

Consider the integral $\displaystyle\int_C dz \exp\left(iaz^n\right)$ where C is the arc in the argand plane ranging from 0 to π/n shown below. We consider the limit as the radius of the arc, R, goes to infinity. Then our integral becomes.



$$\int_C Rd\varphi \exp\left(iaR^n \cdot e^{in\varphi}\right) = \int_0^{\pi/n} Rd\varphi \exp\left[iaR^n\left(\cos n\varphi + i\sin n\varphi\right)\right]$$

$$\leq \int_0^{\pi/n} Rd\varphi \left|\exp\left[iaR^n\left(\cos n\varphi + i\sin n\varphi\right)\right]\right| = \int_0^{\pi/n} Rd\varphi \exp\left[-aR^n \sin n\varphi\right]$$

$$= 2\int_0^{\pi/2n} Rd\varphi \exp\left[-aR^n \sin n\varphi\right] \leq 2\int_0^{\pi/2n} Rd\varphi \exp\left[-aR^n \frac{1}{\pi/2n}\varphi\right]$$

$$= 2\int_0^{\pi/2n} Rd\varphi \exp\left[-\frac{2na}{\pi}R^n\varphi\right] = \frac{\pi R}{naR^n}\left(1-\exp\left[-aR^n\right]\right) \sim \frac{A}{R^{n-1}}$$

Note that we can't extend the arc further because then the sin function becomes negative, which will make the exponential become positive, and will blow up as R goes to infinity. So for the first few values of n, we have

N	ARC	CONVERGENCE
1	π	1
2	π/2	1/R

Note that we also demonstrated these results hold over half the listed arc as well. No cancellations by symmetry have been accomplished here. So these result should hold over any arc probably - I'll discuss this more below, for the n = 1 case.

This indicates that the integral of the exponential over a polynomial from negative inf. to inf. will vanish if the polynomial goes at least as 1/R^a, where a > 0, and that the integral of a squared exponential will go to zero over a quarter contour. This result will be used for the method of steepest decent approximation to integrals. Note that if we double the arc, to run between 0 and π for the n = 2 case, we get

$$\int_C Rd\varphi \exp\left(iaR^2 \cdot e^{i2\varphi}\right) = \int_0^{\pi} Rd\varphi \exp\left[iaR^2\left(\cos 2\varphi + i\sin 2\varphi\right)\right]$$

$$\leq \int_0^{\pi} Rd\varphi \left|\exp\left[iaR^2\left(\cos 2\varphi + i\sin 2\varphi\right)\right]\right| = \int_0^{\pi} Rd\varphi \exp\left[iaR^2 \cos 2\varphi - aR^2 \sin 2\varphi\right]$$

$$\int_0^{\pi} Rd\varphi \left\{\cos\left(aR^2\cos 2\varphi\right) + i\sin\left(aR^2\cos 2\varphi\right)\right\} \exp\left[-aR^2 \sin 2\varphi\right] \sim$$

$$\int_0^{2\pi} Rd\varphi \left\{\cos\left(aR^2\cos\varphi\right) + i\sin\left(aR^2\cos\varphi\right)\right\} \exp\left[-aR^2 \sin\varphi\right] = 0$$

This is a general result, if we integrate over any multiple of these arcs, we get zero, because every even multiple will cause ϕ to run over the entire Argand plane - which must give zero since there are no singularities, and every odd multiple reduces to the case already presented. ACTUALLY, DON'T THINK THIS IS TRUE.

Consider now the integral of a simple exponential over any arc in the upper half-plane, where it goes to 0.

$$\int_C dz e^{itz} = \int_C dz \exp\left[it\,\mathrm{Re}^{i\phi}\right] = \int_{\phi_1}^{\phi_2} Rd\phi \exp\left[it\,\mathrm{Re}^{i\phi}\right] = R\int_{\phi_1}^{\phi_2} d\phi \exp\left[it\,\mathrm{R}(\cos\phi + i\sin\phi)\right]$$
$$= R\int_{\phi_1}^{\phi_2} d\phi \exp\left[it\,\mathrm{R}\cos\phi - tR\sin\phi\right] < R\int_{\phi_1}^{\phi_2} d\phi \exp\left[-tR\sin\phi\right]$$

Now we consider the fact that for any two angles between 0 and π, we have,

$$\sin\phi > \sin(\phi_1) + \frac{\sin(\phi_2) - \sin(\phi_1)}{\phi_2 - \phi_1}\left[\phi - \phi_1\right] = a + m(\phi - \phi_1)$$

So we may replace $\sin\phi$ with this function.

$$\int_C dz e^{itz} < R\int_{\phi_1}^{\phi_2} d\phi \exp\left[-tR\left\{a + m(\phi - \phi_1)\right\}\right] = R\,e^{-tR(a - m\phi_1)}\int_{\phi_1}^{\phi_2} d\phi \exp\left[-tRm\,\phi\right] = -\frac{R\,e^{-tR(a-m\phi_1)}}{tRm}\left[e^{-tRm\phi_1} - e^{-tRm\phi_2}\right]$$
$$< -\frac{e^{-tRa}}{tm}\left[1 - e^{-tRm(\phi_2 - \phi_1)}\right]$$

Now note that the product $m(\phi 2 - \phi 1)$ is always positive, and 'a' is also always positive. So these are decaying exponentials, necessarily for ϕ in the u.h.p. Note also that only if a = 0 (i.e. $\phi 1 = 0$) is it the case that the integral goes as

a constant. Otherwise it exponentially decays. So we have the result that for any arc in the appropriate half plane, the exponential integral goes at least as a constant.

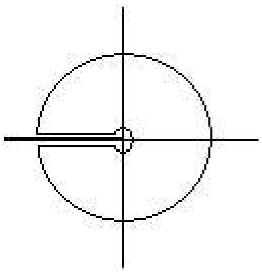
Note, however, that if we include distributions in our allowed results, we can actually integrate a trig function times a rational polynomial of order 1. Similarly, if we care to include derivatives of distributions, I suppose we can integrate any order rational polynomial against a trig function. However, I'm not sure that in these cases, we can use the trick of computing sign integrals by equating it to the imaginary part of the complex exponential. We might have to write the sin function (or cosine) in terms of complex exponentials themselves.

$$\int_{-\infty}^{\infty} dx\, e^{ikx}\frac{x^2 - 1}{x^2 + 1} = \int_{-\infty}^{\infty} dx\, e^{ikx}\left\{1 - \frac{2}{x^2 + 1}\right\} = 2\pi\delta(k) - 2\pi e^{-|k|}$$

Such integrals can often also be determined through use of residues at infinity. If the limit of the rational function is the same on both ends, then this technique should be efficacious. This ought to be reflected in the distributional technique mentioned above in that in such cases, the delta functions will cancel out.

G.3 General Technique for Evaluating integrals from 0 to infinity

We want to evaluate integrals of the following form $\int_0^\infty dx\, f(x)$



Consider the contour to the left which we'll use to evaluate the integral $\int dz\, f(-z)\ln(z)$

We'll use the branch cut shown, although, if desired, we could've used the typical branch cut extending from 0 to infinity, and you can see that the same result would be achieved. Under suitable conditions the integral along the circular contours around the origin and around infinity will go to zero and we'll be left with.

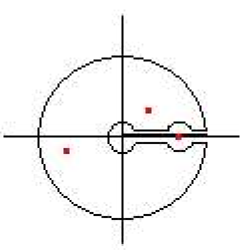
$$\int_{-\infty}^{0} dx\left[\ln(-x)+\pi i\right]f(-x)+\int_{0}^{-\infty} dx\left[\ln(-x)-\pi i\right]f(-x)=2\pi i\sum \mathrm{Res}\left[\ln(z)f(\text{-}z)\right]$$

$$-\int_{0}^{-\infty} dx\left[\ln(-x)+\pi i\right]f(-x)+\int_{0}^{-\infty} dx\left[\ln(-x)-\pi i\right]f(-x)=2\pi i\sum \mathrm{Res}\left[\ln(z)f(\text{-}z)\right]$$

$$-\pi i\int_{0}^{-\infty} dxf(-x)-\pi i\int_{0}^{-\infty} dx\, f(-x)=2\pi i\sum \mathrm{Res}\left[\ln(z)f(\text{-}z)\right]$$

$$2\pi i\int_{0}^{\infty} dxf(x)=2\pi i\sum \mathrm{Res}\left[\ln(z)f(\text{-}z)\right]$$

$$\int_{0}^{\infty} dxf(x)=\sum \mathrm{Res}\left[\ln(z)f(-z)\right]$$



Now, we'll establish the same result with a different contour, but also generalize it by including possible poles on the real axis. As before, we can count on the integrals along the two inner and outer circular contours to vanish.

$$\int_{\infty}^{0} dx\left[\ln(x)\right]f(x)+\pi i\text{Res}\left[\ln(z)f(z),z=p\right]+\int_{0}^{\infty} dx\left[\ln(x)+2\pi i\right]f(x)+\pi i\text{Res}\left[\ln(z)f(z),z=p\right]=2\pi i\sum\text{Res}\left[\ln(z)f(z)\right]$$

$$-\int_{0}^{\infty} dx\left[\ln(x)\right]f(x)+\pi i\text{Res}\left[\ln(z)f(z),z=p^{+}\right]+\int_{0}^{\infty} dx\left[\ln(x)+2\pi i\right]f(x)+\pi i\text{Res}\left[\ln(z)f(z),z=p^{-}\right]=2\pi i\sum\text{Res}\left[\ln(z)f(z)\right]$$

$$2\pi i\int_{0}^{\infty} dx\, f(x)+\pi i\text{Res}\left[\ln(z)f(z),z=p^{+}\right]+\pi i\text{Res}\left[\ln(z)f(z),z=p^{-}\right]=2\pi i\sum\text{Res}\left[\ln(z)f(z)\right]$$

$$\int_{0}^{\infty} dx\, f(x)+2\pi i\text{Res}\left[\text{Ln}(z)f(z),z=p\right]=\sum\text{Res}\left[\ln(z)f(z)\right]-\text{Res}\left[\ln(z)f(z),z=p\right]$$

$$\int_{0}^{\infty} dx\, f(x)=\sum\text{Res}\left[\ln(z)f(z)\right]-\text{Res}\left[\text{Ln}(z)f(z),z=p\right]$$

Where Ln stands for the principal branch of the natural log function

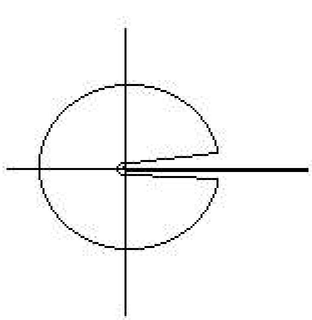
We could also try to modify this formula by changing the contour to the semi-circular one. This would introduce some complications, but might avoid the convergence problems that we run into with the pac - man contour.

Or sometimes its just easier to evaluate: $\int_{0}^{\infty} dx\ln(x)f(x)$ when you want to calculate $\int_{0}^{\infty} dx\, f(x)$ as often times as the change in phase - either from a circular or half circle contour - ought to give you the ln integral you don't care about and a phase times the integral you do care about. That's basically what we're saying above.

OK now let me re-present these ideas in my more intuitive language so that I don't have to memorize the formulas. Suppose that f doesn't have poles on the positive real axis. Consider the integral:

$$\int_{0}^{\infty} dx\, f(x)\rightarrow\int_{0}^{\infty} dx\ln(x)f(x)$$

I'll use the contour below, in which case I'll have:



$$\int_{0}^{\infty} dx\ln(x)f(x)+\int_{\infty}^{0} dx\left[\ln(x)+2\pi i\right]f(x)=2\pi i\sum_{\text{Res}}\ln(x)f(x)$$

$$-2\pi i\int_{0}^{\infty} dx\ln(x)f(x)=2\pi i\sum_{\text{Res}}\ln(x)f(x)$$

$$\int_{0}^{\infty} dx\, f(x)=-\sum_{\text{Res}}\ln(z)f(z)$$

Now assume we want the integral: $\int_a^\infty dx\, f(x) \to \int_a^\infty dx \ln(x-a) f(x)$

Doing the same procedure (and assuming no poles at x > a) we'll have,

$$\int_a^\infty dx \ln(x-a) f(x) + \int_\infty^a dx \left[\ln(x-a) + 2\pi i\right] f(x) = 2\pi i \sum_{\text{Res}} \ln(x-a) f(x)$$
$$-2\pi i \int_a^\infty dx \ln(x-a) f(x) = 2\pi i \sum_{\text{Res}} \ln(x) f(x)$$
$$\int_a^\infty dx\, f(x) = -\sum_{\text{Res}} \ln(z-a) f(z)$$

And now finally consider the integral (note that we seem to assume/require asymptotic integrability of the integrand):

$$\int_a^b dx\, f(x) = \int_a^\infty dx\, f(x) - \int_b^\infty dx\, f(x) \qquad a > b$$

From above we have,

$$\int_a^\infty dx\, f(x) - \int_b^\infty dx\, f(x) = -\sum_{\text{Res}} \ln(x-a) f(x) + \sum_{\text{Res}} \ln(x-b) f(x)$$
$$\int_a^b dx\, f(x) = -\sum_{\text{Res}} \ln\left(\frac{z-a}{z-b}\right) f(z)$$

And so to summarize:

$$\int_0^\infty dx\, f(x) = -\sum_{\text{Res}} \ln(z) f(z)$$
$$\int_a^\infty dx\, f(x) = -\sum_{\text{Res}} \ln(z-a) f(z)$$
$$\int_a^b dx\, f(x) = -\sum_{\text{Res}} \ln\left(\frac{z-a}{z-b}\right) f(z) =$$

One could probably do much the same thing by using a half-circle contour in the upper half or lower half plane. This would be advantageous because there would be fewer residues to sum over.

G.4 Other General Ideas

See *The Cauchy Method of Residues*, by Mitronovich & Keckic for more interesting techniques to evaluate definite integrals. Examples include integrals of the form ...

$$\int_0^{2\pi} dx\, f(x) \ln\left[\sin(x/2)\right]$$

which is an integral encountered in many cases where have to invert the Cauchy kernel when dealing with integral equations. (You have to make a trigonometric change of variables to put it in this form though)

Other integrals discussed are ones that seem to be encountered in the evaluation of Green's functions (specifically spectral functions which are periodic over 2πi), and many others.

Another thing to keep in mind sometimes is that it is often useful to use the identities...

$$\theta(t) = \int \frac{d\omega}{2\pi} \frac{i}{\omega + i0^+} e^{-i\omega t}$$

$$\operatorname{sgn}(t) = \int \frac{d\omega}{\pi} \cdot \frac{1}{\omega} \sin(\omega t)$$

$$e^{-|t|} = \int_0^{\infty} \frac{d\omega}{\pi/2} \frac{\cos(\omega t)}{\omega^2 + 1}$$

$$\frac{1}{\sqrt{L^2 - t^2}} = \int_0^{2\pi} \frac{d\omega}{2\pi} \frac{1}{L + t\sin(\omega)}$$

$$\frac{1}{\sin(\pi t)} = \int_0^{\infty} \frac{d\omega}{\pi} \frac{\omega^{p-1}}{\omega + 1}$$

Thus, if we have |x| in an expression, we can instead use the identity |x| = sgn(x)x, and then insert the integral identity for sgn(x). It is also useful in extending an integral of an odd function from 0 to L to -L to L - thus we can convert an integral from 0 to infinity of an odd function into an integral from 0 to infinity of an even function, which can then be extended to the real line, and a contour probably can be used. This will work sometimes when the general formula above doesn't (because one of the contour integrals doesn't go to zero). If we have a piece - wise continuous function or something, the theta representation might be useful. Finally, if the integral is undefined because of an infinity at the upper or lower ends, these representations might provide a nice way to regulate the infinity, like is done for the theory of Green's functions.

Also it might be of use to use the limit below when trying to evaluate integrals over the semi infinite real line, because you can use either the pac - man or semi - circular contour for the integral on the right.

$$\int_0^{\infty} dx\, f(x) = \lim_{\alpha \to \infty} \int_0^{\infty} dx \frac{f(x)}{x^{1/\alpha}}$$

G.5 Miscellaneous Integrals Evaluated

A. $\displaystyle\int_0^{2\cdot\pi} R(\cos(\phi), \sin(\phi))\, d\phi$

Where R is a rational function involving the two arguments:
The usual procedure is to substitute in the relations

Convert sines and cosines to complex number representation on unit circle.

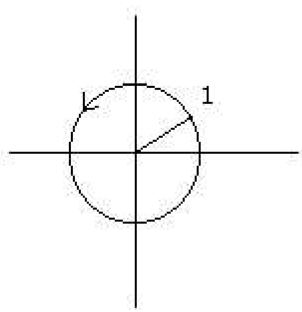
$$\sin(\phi) := \frac{e^{i\cdot\phi} - e^{-i\cdot\phi}}{2\cdot i} \qquad \cos(\phi) := \frac{e^{i\cdot\phi} + e^{-i\cdot\phi}}{2}$$

$$\sin(\phi) := \frac{z - z^{-1}}{2\cdot i} \qquad \cos(\phi) := \frac{z + z^{-1}}{2}$$

In addition, for example $\cos(n\cdot\phi) := \dfrac{z^{n} + z^{-n}}{2}$

$$d\phi := \frac{dz}{i\cdot z}$$

and the line integral from 0 to 2π transforms to a contour integral as shown. All the zeroes of the rational function should lie inside the contour.



For example, let us consider

$$I := \int_{0}^{2\cdot\pi} \frac{1}{a + b\cdot\sin(\phi)}\, d\phi \qquad a > b$$

$$I := \int \frac{\frac{1}{i\cdot z}}{a + b\cdot\frac{z - z^{-1}}{2\cdot i}}\, dz$$

Where the integration is along the unit circle

$$I := \int \frac{1}{\frac{b}{2}\cdot z^2 + a\cdot i\cdot z - \frac{b}{2}}\, dz$$

$$I := \int \frac{\frac{2}{b}}{\left[z - \left(\frac{\sqrt{a^2 - b^2} - a}{b}\cdot i\right)\right]\cdot\left[z - \left(\frac{-\sqrt{a^2 - b^2} - a}{b}\cdot i\right)\right]\cdot\frac{b}{2}}\, dz$$

Note you can't use the residue at inf. b/c its zero.

Only the first root is inside the circle. The integral is just $2\cdot\pi\cdot i\cdot\sum_n \mathrm{Res}(w(z), z_n)$ so

$$I := 2\cdot\pi\cdot i\cdot\frac{2}{b}\cdot\left(\frac{\sqrt{a^2 - b^2} - a}{b}\cdot i - \frac{-\sqrt{a^2 - b^2} - a}{b}\cdot i\right)^{-1}$$

$$I := \frac{2\cdot\pi}{\sqrt{a^2 - b^2}}$$

B. $\int_0^\infty dx \frac{\ln(x)}{x^2+4}$

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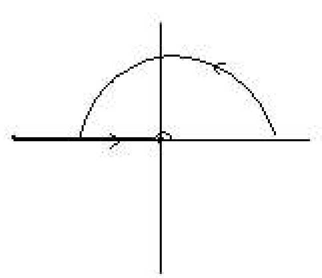
To proceed we will evaluate the integrand along the contour shown



$$\int_C dz \frac{\ln(z)}{z^2+4} = \int_{-\infty}^{-\varepsilon} dz \frac{\ln(z)}{z^2+4} + \int_{|z|=\varepsilon} dz \frac{\ln(z)}{z^2+4} + \int_{\varepsilon}^{\infty} \frac{\ln(z)}{z^2+4} + \int_{|z|=R} dz \frac{\ln(z)}{z^2+4} = 2\pi i \mathrm{Res}\left[\frac{\ln(z)}{z^2+4}, z = 2i\right]$$

$$\int_{-\infty}^{-\varepsilon} dz \frac{\ln(z)}{z^2+4} = \int_{-\infty}^{-\varepsilon} dx \frac{\ln(-x)+i\pi}{x^2+4} = \int_{\varepsilon}^{\infty} dx \frac{\ln(x)+i\pi}{x^2+4}$$

$$\int_{|z|=\varepsilon} dz\frac{\ln(z)}{z^2+4} = i\int_{|z|=\varepsilon} \varepsilon d\phi\frac{\ln(\varepsilon)+i\phi}{\varepsilon^2+4} = i\int_{|z|=\varepsilon} d\phi\frac{\varepsilon\ln(\varepsilon)+\varepsilon i\phi}{\varepsilon^2+4} \to i\int_{|z|=\varepsilon} d\phi\frac{\varepsilon+\varepsilon i\phi}{\varepsilon^2+4} \to 0$$

$$\int_{|z|=R} dz\frac{\ln(z)}{z^2+4} = \int_{|z|=R} Rd\phi\frac{\ln(R)+i\phi}{R^2+4} \to \int_{|z|=R} d\phi\frac{\ln(R)+i\phi}{R} \to 0$$

$$\mathrm{Res}\left[\frac{\ln(z)}{z^2+4}, z=2i\right] = \mathrm{Res}\left[\frac{\ln(2e^{\pi i/2})}{(z+2i)(z-2i)}, z=2i\right] = \frac{\ln(2)+\pi i/2}{4i}$$

Thus, equating the real and imaginary parts, we have...

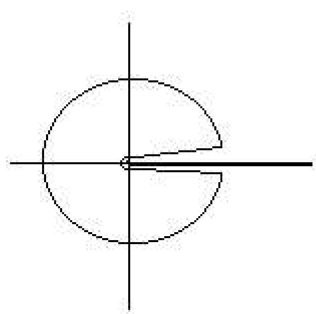
$$\int_{\varepsilon}^{\infty} dx\frac{2\ln(x)+i\pi}{x^2+4} = 2\pi i\frac{\ln(2)+\pi i/2}{4i} = \frac{\pi\ln(2)}{2}+\frac{\pi^2}{4}i$$

$$\int_{\varepsilon}^{\infty} dx\frac{\ln(x)}{x^2+4} = \frac{\pi\ln(2)}{4}$$

$$\int_{\varepsilon}^{\infty} dx\frac{1}{x^2+4} = \frac{\pi}{4}$$

c. $$\int_0^{\infty} dx\frac{1}{x^{1/\alpha}(x+1)}$$

Let us use the contour to the left to evaluate the integral.



$$\int_{|z|=\varepsilon}\frac{1}{z^{1/\alpha}(z+1)} + \int_0^{\infty} dx\frac{1}{x^{1/\alpha}(x+1)} + \int_{|z|=R}\frac{1}{z^{1/\alpha}(z+1)} + \int_{\infty}^{0} dz\frac{1}{z^{1/\alpha}(z+1)} = 2\pi i\mathrm{Res}\left[\frac{1}{z^{1/\alpha}(z+1)}, z=-1\right]$$

$$\int_{|z|=\varepsilon} \frac{1}{z^{1/\alpha}(z+1)} = \int_{|z|=\varepsilon} id\phi \frac{\varepsilon}{\varepsilon^{1/\alpha} e^{i\phi/\alpha}(\varepsilon e^{i\phi/\alpha}+1)} \le \int_{|z|=\varepsilon} id\phi \frac{\varepsilon(1-\varepsilon e^{-i\phi/\alpha})}{\varepsilon^{1/\alpha}(1+\varepsilon^2)} \to 0$$

$$\int_{|z|=R} id\phi \frac{R}{R^{1/\alpha} e^{i\phi/\alpha}(Re^{i\phi/\alpha}+1)} \le \int_{|z|=R} id\phi \frac{R(1-Re^{-i\phi/\alpha})}{R^{1/\alpha}(1+R^2)} \to 0$$

$$\int_{\infty}^{0} dz \frac{1}{z^{1/\alpha}(z+1)} = \int_{\infty}^{0} dx \frac{1}{x^{1/\alpha} e^{2\pi i/\alpha}(x+1)} = -e^{-2\pi i/\alpha} \int_{0}^{\infty} dx \frac{1}{x^{1/\alpha}(x+1)}$$

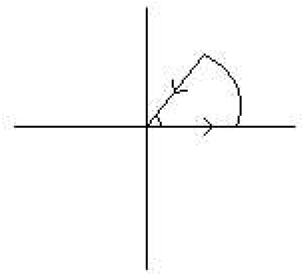
$$\operatorname{Res}\left[\frac{1}{z^{1/\alpha}(z+1)}, z=-1\right] = \frac{1}{e^{\pi i/\alpha}}$$

Thus we have, when we put them all together, that

$$(1-e^{-2\pi i/\alpha})\int_{0}^{\infty} dx \frac{1}{x^{1/\alpha}(x+1)} = \frac{2\pi i}{e^{\pi i/\alpha}}$$

$$\int_{0}^{\infty} dx \frac{1}{x^{1/\alpha}(x+1)} = 2\pi i \frac{e^{-\pi i/\alpha}}{1-e^{-2\pi i/\alpha}} = \pi \frac{2i}{e^{\pi i/\alpha} - e^{-\pi i/\alpha}} = \pi \csc(\pi/\alpha)$$

D. $\displaystyle\int_0^\infty \frac{x^m}{x^n+1}\,dx$ where m and n are integers and the angle shown in the contour is 2π/n. That way you won't have too many residues to evaluate.



$$\int_0^\infty dx\frac{x^m}{x^n+1}+\int_{arc} dz\frac{z^m}{z^n+1}+\int_{diag.} dz\frac{z^m}{z^n+1}=2\pi i\mathrm{Res}\left[\frac{z^m}{z^n+1},z=(-1)^{1/n}\right]$$

$$\int_{arc} dz\frac{z^m}{z^n+1}=\int_\infty^0 dxe^{2\pi i/n}\frac{x^m e^{2\pi mi/n}}{x^n+1}=-e^{2\pi(m+1)i/n}\int_0^\infty dx\frac{x^m}{x^n+1}$$

$$\mathrm{Res}\left[\frac{z^m}{z^n+1},(-1)^{1/n}\right]=\mathrm{Res}\left[\frac{z^m}{z^n+1},e^{\pi i/n}\right]=\frac{e^{\pi mi/n}}{ne^{\pi i(n-1)/n}}=\frac{1}{n}e^{(m+1)\pi i/n}$$

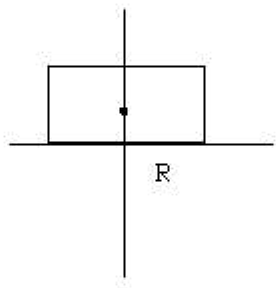
Now combining all of our terms, we come to

$$(1-e^{2\pi(m+1)i/n})\int_0^\infty dx\frac{x^m}{x^n+1}=2\pi i\frac{1}{n}e^{\pi(m+1)i/n}$$

$$\int_0^\infty dx\frac{x^m}{x^n+1}=\frac{2\pi i}{n}\frac{e^{\pi(m+1)i/n}}{1-e^{2\pi(m+1)i/n}}=\frac{2\pi i}{n}\frac{1}{e^{-(m+1)\pi i/n}-e^{(m+1)\pi i/n}}=-\frac{\pi}{n}\csc\left[\pi\left(\mathrm{m}+1\right)/\mathrm{n}\right]$$

I don't know why I am getting that anomolous negative sign - something I'll have to look into later

E. $\int_{-\infty}^{\infty}\frac{e^x}{1+e^{a\cdot x}}dx$ $a>1$



When the integrand has a cosh function in the denominator. The middle of the contour should be $\pi/2$, and it should go through π, while r goes to infinity. A similar scheme can be devised if sinh is in the denominator.

Look at $\displaystyle\int_C \frac{e^z}{1+e^{a\cdot z}}\,dz$ equals $\displaystyle\int_C \frac{e^x\cdot(\cos(y)+i\cdot\sin(y))}{1+e^{a\cdot x}\cdot(\cos(a\cdot y)+i\cdot\sin(a\cdot y))}\,dz$

Let the height of the rectangle equal $2\pi/a$; parameterize the ends of the rectangle by $x(t) = R$, $y(t) = t$, t between 0 and $2\pi/a$.

$$\int_{-\infty}^{\infty}\frac{e^x}{1+e^{a\cdot x}}\,dx-\left(\int_{-\infty}^{\infty}\frac{e^x}{1+e^{a\cdot x}}\,dx\right)\cdot\left(\cos\left(2\cdot\frac{\pi}{a}\right)+i\cdot\sin\left(2\cdot\frac{\pi}{a}\right)\right)+\int_0^{2\cdot\frac{\pi}{a}}\frac{e^R\cdot(\cos(t)+i\cdot\sin(t))}{1+e^{a\cdot R}\cdot(\cos(a\cdot t)+i\cdot\sin(a\cdot t))}\cdot i\,dt-\int_0^{2\cdot\frac{\pi}{a}}\frac{e^{-R}\cdot(\cos(t)+i\cdot\sin(t))}{1+e^{-a\cdot R}\cdot(\cos(a\cdot t)+i\cdot\sin(a\cdot t))}\cdot i\,dt$$

Using the ML inequality, you can show that the last two integrals vanish to 0, as R goes to inf.

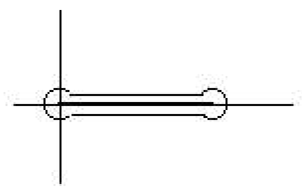
$\dfrac{e^x}{1+e^{a\cdot x}}$ has a first order pole at $i\pi/a$, which you can verify by taking the limit $\displaystyle\lim_{z\to i\cdot\frac{\pi}{a}}\frac{e^z\cdot\left(z-i\cdot\frac{\pi}{a}\right)}{1+e^{a\cdot z}}$

Use L'Hospitals Rule; this limit is also the residue and equals $-(\cos(\pi/a) + i\sin(\pi/a))/a$. Then multiply by $2\pi i$, and equate the imaginary parts of both sides to get.

$\displaystyle\int_{-\infty}^{\infty}\frac{e^x}{1+e^{a\cdot x}}\,dx := 2\cdot\pi\cdot\frac{\cos\left(\frac{\pi}{a}\right)}{a\cdot\sin\left(2\cdot\frac{\pi}{a}\right)}$ which equals $\displaystyle\frac{\pi}{a\cdot\sin\left(\frac{\pi}{a}\right)}$

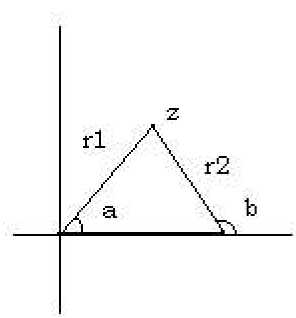
F. $\displaystyle\int_0^1 \frac{\sqrt{x\cdot(1-x)}}{1+x^2}\,dx$

When given something that has two branch points at -1 and 1 or at just about any two points in the plane is given, a branch cut like the one shown can be made, and contour like the one shown can be integrated over.



$z := r_1\cdot e^{i\cdot(a+2\cdot\pi\cdot k)}$ $\quad 1-z := -r_2\cdot e^{i\cdot(b+2\cdot\pi\cdot m)}$ equals $r_2\cdot e^{i\cdot(b+\pi+2\cdot\pi\cdot m)}$

$$\sqrt{z\cdot(1-z)} := \sqrt{r_1\cdot r_2}\cdot e^{i\cdot\frac{[a+b+2\cdot\pi\cdot(k+m)+\pi]}{2}}$$



Now points along the line a = 0, b = π, must be real, so make (a + b + π + 2π(k+m))/2 = π(k+m+1) equal to a multiple of 2π, say 0. Therefore k + m = -1. And you could say that k = 0, m = -1, that is, we start angle a off at 0, and angle b off at -2π along the positive x axis.

so $\quad \sqrt{z\cdot(1-z)} := \sqrt{r_1\cdot r_2}\cdot e^{i\cdot\frac{(a+b-\pi)}{2}}$

On the back side, under the cut, a = 0 b = -π, and a phase factor of -1 will be picked up, but since the underside will be integrated backwards, you'll get I again.

Integral along the circular regions will go to 0, since the integrand will go to zero there.

$$\int_0^1 \frac{\sqrt{x\cdot(1-x)}}{1+x^2}\,dx := \frac{1}{2}\cdot 2\cdot\pi\cdot i\cdot\sum_n \mathrm{Res}\left[\frac{\sqrt{z\cdot(1-z)}}{1+z^2}\right]$$

Residues are those outside the contour, and occur at z = +/- i, and at inf.

At -i, a = -π/2 (or 3π/2, etc.) and b = -3π/4 (or 5π/4, etc.), which implies the residue is $\dfrac{e^{i\cdot\frac{-9\cdot\pi}{8}}\cdot\sqrt{\sqrt{2}}}{-2\cdot i}$ equals

$$\frac{\left(-\cos\left(\frac{\pi}{8}\right)+i\cdot\sin\left(\frac{\pi}{8}\right)\right)\cdot\sqrt{\sqrt{2}}}{-2\cdot i}$$

Note that it doesn't matter how you represent the 1+z^2 = 2i since its only raised to an integer power and will come out the same anyway.

At i, a = π/2, b = 3π/4, and the Res will equal $\dfrac{e^{i\cdot\frac{\pi}{8}}\cdot\sqrt{\sqrt{2}}}{2\cdot i}$ equals $\dfrac{\left(\cos\left(\frac{\pi}{8}\right)+i\cdot\sin\left(\frac{\pi}{8}\right)\right)\cdot\sqrt{\sqrt{2}}}{2\cdot i}$

The sum equals $\dfrac{\sqrt{\sqrt{2}}\cdot\cos\left(\frac{\pi}{8}\right)}{i}$

Residue at inf. = -Res(f(1/w)/w^2,0)

Note procedure can be generalized to - change variables to z = 1/w and divide by w^2, take the nec. derivatives, and set up the limit to be taken. Then change variables back to z, and do this in the proper branch.

Look at $\dfrac{\sqrt{\frac{1}{w}\cdot\left(1-\frac{1}{w}\right)}}{\left[1+\left(\frac{1}{w}\right)^2\right]\cdot w^2}$ equals $\dfrac{\sqrt{\frac{w-1}{w^2}}}{w^2+1}$ equals $\dfrac{\sqrt{w-1}}{w\cdot\left(1+w^2\right)}$

$$\mathrm{Res} := \lim_{w\to 0}\frac{-\sqrt{w-1}}{\left(1+w^2\right)}$$ equals $\lim_{w\to 0} -\sqrt{w-1}$ but in order to be able to tell which branch it is in, make the substitution back to z.

$$\mathrm{Res} := \lim_{z\to\infty}\frac{-\sqrt{z\cdot(1-z)}}{z} \qquad \frac{-\sqrt{r_1\cdot r_2}\cdot e^{i\cdot\frac{(a+b-\pi)}{2}}}{r_1\cdot e^{i\cdot(a+0)}}$$

as z goes to inf. a goes to b (i.e. the rays are approximately parallel), and r1 goes to r2.

Therefore the Res equals $-e^{-i\cdot\frac{\pi}{2}}$ equals i

So $\quad I := \pi \cdot i \cdot \left(i + \frac{\sqrt{\sqrt{2}} \cdot \cos\left(\frac{\pi}{8}\right)}{i} \right)$ $\quad$ equals $\quad \pi \cdot \left(\sqrt{\sqrt{2}} \cdot \cos\left(\frac{\pi}{8}\right) - 1 \right)$

To evaluate the residue at infinity, it might be more straightforward to expand the term in a Taylor series valid around infinity - as follows.

$$\frac{\sqrt{z(1-z)}}{1+z^2} = \frac{\sqrt{-z^2(1-1/z)}}{1+z^2} = \frac{-zi\sqrt{1-1/z}}{z^2(1+1/z^2)} = -\frac{i}{z}\left(1-\frac{1}{2z}-\frac{1}{8z^2}-...\right)\left(1-\frac{1}{z^2}+\frac{1}{z^4}+...\right)$$
$$= -i\left(\frac{1}{z}-\frac{1}{2z^2}-\frac{1}{8z^3}-...\right)\left(1-\frac{1}{z^2}+\frac{1}{z^4}+...\right)$$

Thus the residue is clearly i (remember that the definition of the residue at infinity has that minus sign in it). Note that I used sqrt(-1) = -i. This was necessary to stay in the same branch. For instance, if you plug 2 into the expession before any manipulations you get -sqrt(2)i. To get the same after the manipulations, you have to use sqrt(-1)=-i. So, in general, after any step that doesn't preserve the branch automatically, you'll want to check that you made the right choices for sqrts and everything, so that you can be certain you're in the same branch.

So the general idea here for evaluating the residue at inf. is to get an expression for the residue using the w substitution, simplify as much as possible without changing the branch you're in; now substitute back to z (note that this won't give you your original function because in addition to making the 1/w substitution, you divided by w^2) and evaluate the limit in the branch you're in.

G. $\quad \int_{-\infty}^{\infty} dx \frac{x}{x^2+x+1}$

we can evaluate the Cauchy principle value of this integral - the integral in the ordinary sense doesn't even exist of course. We will use the semi - circular contour as usual. The principal value by the way is the value you would get if you integrated from - L to L and then took the limit as L goes to infinity. This is the same idea as the P.V. for singularities at finite points.

Let us write out the integration along the contour...

$$\int_{-\infty}^{\infty} dx \frac{x}{x^2+x+1} + \int_{|z|=R} dz \frac{z}{z^2+z+1} = 2\pi i \text{Res}\left[\frac{z}{z^2+z+1}, z = \frac{\sqrt{3}i-1}{2}\right]$$

$$\int_{|z|=R} dz \frac{z}{z^2+z+1} = -\pi i \text{Res}\left[\frac{z}{z^2+z+1}, z = \infty\right] = \pi i$$

$$\text{Res}\left[\frac{z}{z^2+z+1}, z=\frac{\sqrt{3}i-1}{2}\right]=\frac{\frac{\sqrt{3}i-1}{2}}{2\left(\frac{\sqrt{3}i-1}{2}\right)+1}=\frac{1}{2}\frac{\sqrt{3}i-1}{\sqrt{3}i}$$

Therefore we have...

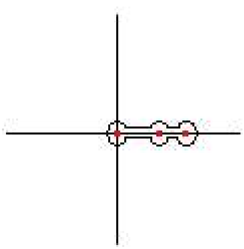
$$\int_{-\infty}^{\infty} dx\frac{x}{x^2+x+1}=2\pi i\frac{1}{2}\frac{\sqrt{3}i-1}{\sqrt{3}i}-\pi i=\pi i\left(1-\frac{1}{\sqrt{3}i}\right)-\pi i=-\frac{\pi}{\sqrt{3}}$$

the minus sign is supposed to be there

Note that normally to evaluate integrals we require that the integrand at infinity go to zero faster than 1/R so that when we multiply by R (from the measure) the integral along the arc contour goes to zero. But that is too stringent. If the integrand goes as 1/R, then we can still evaluate the integral along the arc contour - it is just half the residue at infinity. But if the integrand goes to zero slower than 1/R, we will have a problem - but only if the arc - contour is a semi circle. If it is a whole circle then that integral is just the residue.

H. $\int_0^1 dz\frac{z\sqrt{z(1-z)}}{z-u}$

Now let us consider an integral similar to the one above, but with a singularity on the branch cut



We can see that the integral along the points z = 0, and z = 1 will vanish as before, so evaluating the integrand along the contour, we have. Integrating along the underside will pick up the same quantity, except with a minus sign, which will be reversed when we switch the order of integration - as before. So we end up with...

$$2\int_0^1 dz\frac{z\sqrt{z(1-z)}}{z-u}=2\pi i\text{Res}\left[\frac{z\sqrt{z(1-z)}}{z-u}, z=\infty\right]+\pi i\text{Res}\left[\frac{z\sqrt{z(1-z)}}{z-u}, z=u^+\right]+\pi i\text{Res}\left[\frac{z\sqrt{z(1-z)}}{z-u}, z=u^-\right]$$

where the +/- stands for u on the top/bottom contour.

The residue on the top side is equal to

$$\mathrm{Res}\left[\frac{z\sqrt{z(1-z)}}{z-u}, z=u^{+}\right]=u\sqrt{u(1-u)}$$

The residue on the bottom side is equal to

$$\mathrm{Res}\left[\frac{z\sqrt{z(1-z)}}{z-u}, z=u^{-}\right]=-u\sqrt{u(1-u)}$$ since when we go along the bottom contour, we pick up a minus sign

Thus the top and bottom residues cancel eachother out. We are left with the residue at infinity.

$$\mathrm{Res}\left[\frac{z\sqrt{z(1-z)}}{z-u}, z=\infty\right]=-\mathrm{Res}\left[\frac{1}{w^2}\frac{(1/w)\sqrt{(1/w)(1-(1/w)}}{(1/w)-u}, w=0\right]=-\mathrm{Res}\left[\frac{1}{w^2}\frac{1}{1-uw}\sqrt{\frac{1}{w}\left(1-\frac{1}{w}\right)}, w=0\right]$$

$$=-\mathrm{Res}\left[\frac{1}{w^2}\frac{1}{1-uw}\sqrt{\frac{w-1}{w^2}}, w=0\right]=-\mathrm{Res}\left[\frac{1}{w^3}\frac{\sqrt{w-1}}{1-uw}, w=0\right]$$

$$=-\frac{1}{2}\lim_{w\to 0}\frac{\partial^2}{\partial w^2}\frac{\sqrt{w-1}}{1-uw}=-\frac{1}{2}\lim_{w\to 0}\frac{\partial}{\partial w}\frac{\sqrt{w-1}}{1-uw}\left\{\frac{1}{2(w-1)}+\frac{u}{1-uw}\right\}$$

$$=-\frac{1}{2}\lim_{w\to 0}\frac{\sqrt{w-1}}{1-uw}\left\{\frac{1}{2(w-1)}+\frac{u}{1-uw}\right\}\left\{\frac{1}{2(w-1)}+\frac{u}{1-uw}\right\}+\frac{\sqrt{w-1}}{1-uw}\left\{-\frac{1}{2(w-1)^2}+\frac{u^2}{(1-uw)^2}\right\}$$

$$=-\frac{1}{2}\lim_{w\to 0}\sqrt{w-1}\left\{-\frac{1}{2}+u\right\}\left\{-\frac{1}{2}+u\right\}+\sqrt{w-1}\left\{-\frac{1}{2}+u^2\right\}$$

$$=-\frac{1}{2}\left\{\left(-\frac{1}{2}+u\right)^2+\left(-\frac{1}{2}+u^2\right)\right\}\lim_{w\to 0}\sqrt{w-1}$$

Now recalling the result for the limit of the square root quantity that was previously taken above -look at the similar contour integral done previously

$$\mathrm{Res}\left[\frac{z\sqrt{z(1-z)}}{z-u}, z=\infty\right]=\frac{i}{2}\left\{\left(-\frac{1}{2}+u\right)^2+\left(-\frac{1}{2}+u^2\right)\right\}$$

This residue can be more easily calculated by expanding at infinity.

$$\frac{z\sqrt{z(1-z)}}{z-u}=\frac{z\sqrt{(-1)(1-1/z)}}{1-u/z}=\frac{-zi\sqrt{1-1/z}}{1-u/z}\approx -zi\left(1+\frac{u}{z}-\frac{u^2}{z^2}\right)\left(1-\frac{1}{2z}-\frac{1}{8z^2}\right)=-i\left(z+u-\frac{u^2}{z}\right)\left(1-\frac{1}{2z}-\frac{1}{8z^2}\right)$$

By picking out the relevant terms, we can see that the residue is $i\left(u^2-\frac{u}{2}-\frac{1}{8}\right)$

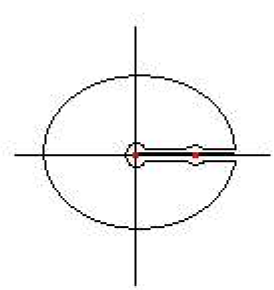
where we also remember the negative sign in the definition of the residue at infinity.

Thus we have

$$\int_0^1 dz\frac{z\sqrt{z(1-z)}}{z-u}=\pi i\mathrm{Res}\left[\frac{z\sqrt{z(1-z)}}{z-u},z=\infty\right]=-\frac{\pi}{2}\left\{\left(-\frac{1}{2}+u\right)^2+\left(-\frac{1}{2}+u^2\right)\right\}=-\frac{\pi}{2}\left\{u^2-u+\frac{1}{4}+u^2-\frac{1}{2}\right\}$$
$$=-\frac{\pi}{2}\left\{2u^2-u-\frac{1}{4}\right\}$$

According to the notepad results, this formula is accurate!

I. $\int_0^\infty dy\frac{x+y}{x-y}\frac{1}{\sqrt{xy}}\frac{1}{1+y^2}$



Now let us consider another integral with a branch cut that goes through a pole. I believe that these can be done in the normal way. The semicircular integral around the pole in the positive Im axis is simply +/-πiRes[function, z = pole], where the function is the one evaluated in the positive Im axis plane. The semicircular integral around the pole in the negative Im axis is simply +/-πiRes[function, z = pole], where the function is the one evaluated in the negative Im axis plane. So I think you can evaluate this with residues. We are just restricted to semicircular contours - because of the discontinuity, and have to evaluate the function on both sides of the pole according to the branch cut. For example...

Extending this to the complex plane and evaluating this around the contour illustrated, we have 0 for the large circular arc, and the small circular arc around the pole at 0. This is fairly obvious as we can see that the integrand goes as 1/R^(3/2) for large y (recall the factor of R from the measure. It also goes as ε^(1/2) for small y. Thus they both go to zero.
The integral back along the contour is equal to

$$\int_\infty^0 dy\frac{x+y}{x-y}\frac{1}{\sqrt{xye^{2\pi i}}}\frac{1}{1+y^2}=-\int_\infty^0 dy\frac{x+y}{x-y}\frac{1}{\sqrt{xy}}\frac{1}{1+y^2}=\int_0^\infty dy\frac{x+y}{x-y}\frac{1}{\sqrt{xy}}\frac{1}{1+y^2}$$

The evaluation of the contour around the singularity at x.

$$\int_{pole} dz \frac{x+z}{x-z}\frac{1}{\sqrt{xz}}\frac{1}{1+z^2} = -\pi i \text{Res}\left[\frac{x+z}{x-z}\frac{1}{\sqrt{xz}}\frac{1}{1+z^2}, z=x\right] - \pi i \text{Res}\left[\frac{x+z}{x-z}\frac{1}{\sqrt{xze^{2\pi i}}}\frac{1}{1+z^2}, z=x\right]$$

$$= -\pi i \text{Res}\left[\frac{x+z}{x-z}\frac{1}{\sqrt{xz}}\frac{1}{1+z^2}, z=x\right] + \pi i \text{Res}\left[\frac{x+z}{x-z}\frac{1}{\sqrt{xz}}\frac{1}{1+z^2}, z=x\right]$$

$$= 0$$

This accounts for the counterintuitive fact that when you have a branch cut running through a pole, you can sometimes ignore it. The factor of two that seems to be missing below comes from the fact that we're dividing by two since the integral backwards along the contour reproduced our integral to be calculated.

$$\int_0^\infty \frac{x+y}{x-y}\frac{1}{\sqrt{xy}}\frac{1}{1+y^2} = \pi i \text{Res}\left[\frac{x+y}{x-y}\frac{1}{\sqrt{xy}}\frac{1}{1+y^2}, y=i\right] + \pi i \text{Res}\left[\frac{x+y}{x-y}\frac{1}{\sqrt{xy}}\frac{1}{1+y^2}, y=-i\right]$$

$$= \pi i\left[\frac{x+i}{x-i}\frac{1}{\sqrt{xi}}\frac{1}{2i}\right] + \pi i\left[\frac{x-i}{x+i}\frac{1}{\sqrt{x\cdot -i}}\frac{1}{-2i}\right]$$

$$= \frac{\pi}{2}\left[\frac{x+i}{x-i}\frac{1}{\sqrt{xe^{\pi i/2}}}\right] - \frac{\pi}{2}\left[\frac{x-i}{x+i}\frac{1}{\sqrt{xe^{3\pi i/2}}}\right]$$

$$= \frac{\pi}{2\sqrt{x}}\left[\frac{(x+i)^2}{x^2+1}e^{-\pi i/4}\right] - \frac{\pi}{2\sqrt{x}}\left[\frac{(x-i)^2}{x^2+1}e^{-3\pi i/4}\right]$$

$$= \frac{\pi}{2\sqrt{2x}\left(x^2+1\right)}\left[(x+i)^2(1-i)\right] - \frac{\pi}{2\sqrt{2x}\left(x^2+1\right)}\left[(x-i)^2(-1-i)\right]$$

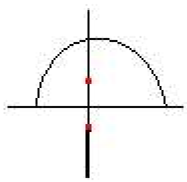
$$= \frac{\pi}{2\sqrt{2x}\left(x^2+1\right)}\left[\left(x^2+2ix-1\right)(1-i) + \left(x^2-2xi-1\right)(1+i)\right]$$

$$= \frac{\pi}{2\sqrt{2x}\left(x^2+1\right)}\left[x^2+2ix-1-ix^2+2x+i+x^2-2xi-1+ix^2+2x-i\right]$$

$$= \frac{\pi}{\sqrt{2x}\left(x^2+1\right)}\left[x^2+2x-1\right]$$

J. $\int_0^\infty dy \frac{\ln(1+x^2)}{1+x^2}$

Consider this integral by calculating the ancillary integral $\int\limits_C dz \frac{\ln(z+i)}{1+z^2}$ along the contour shown



Recognizing that

$$\int\limits_0^\infty dx \ln\left[1+x^2\right] = \int\limits_0^\infty dx \ln\left(i+x\right) + \int\limits_0^\infty dx \ln(x-i) = \int\limits_0^\infty dx \ln\left(i+x\right) + \int\limits_{-\infty}^0 dx \ln(-x-i)$$

$$= \int\limits_0^\infty dx \ln\left(i+x\right) + \int\limits_{-\infty}^0 dx \ln(i+x) + \pi i = \int\limits_{-\infty}^\infty dx \ln\left(i+x\right) + \int\limits_0^\infty dx \pi i$$

In retrospect, I don't think that I can do this - but you get the idea. We are led to consider a new integral

$$\int\limits_{-\infty}^0 dx \frac{\ln(x+i)}{1+x^2} + \int\limits_0^\infty dx \frac{\ln(x+i)}{1+x^2} + \int\limits_{|z|=R} dz \frac{\ln(z+i)}{1+z^2} = 2\pi i \operatorname{Res}\left[\frac{\ln(z+i)}{1+z^2}, z=i\right]$$

$$\int\limits_0^\infty dx \frac{\ln(-x+i)}{1+x^2} + \int\limits_0^\infty dx \frac{\ln(x+i)}{1+x^2} = 2\pi i \frac{\ln(2i)}{2i}$$

$$\int\limits_0^\infty dx \frac{\ln\left(-x^2-1\right)}{1+x^2} = \pi \ln(2i)$$

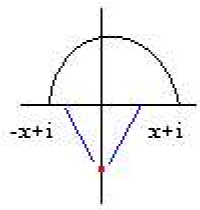
$$\int\limits_0^\infty dx \frac{\ln\left(1+x^2\right)+\pi i}{1+x^2} = \pi \ln(2e^{\pi i/2})$$

$$\int\limits_0^\infty dx \frac{\ln\left(1+x^2\right)+\pi i}{1+x^2} = \pi \ln(2) + \frac{\pi^2}{2}$$

Thus $\int\limits_0^\infty dx \frac{\ln\left(1+x^2\right)}{1+x^2} = \pi \ln(2)$

Note that we explicitly used the properties $\ln(x^2+1)$ is real, $\ln(-x+i)+\ln(x+i) = \ln(x^2+1)+\pi i$

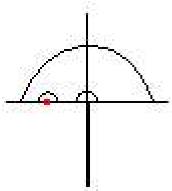
The second one can be verified by forming x + i, and -x + i, from the branch point in the diagram above. We let the angle go from $-\pi/2$ to $3\pi/2$ in the CCW direction and then calculating both sides bears out the equality. The first one is implicit in the second one.



$$\ln(-x+i)+\ln(x+i)$$
$$=\ln\sqrt{x^2+1}+i\tan^{-1}(1/x)+\ln\sqrt{x^2+1}+i\left[\pi-\tan^{-1}(1/x)\right]$$
$$=\ln\left(x^2+1\right)+i\pi$$

K. $\displaystyle\int_0^\infty dx\frac{1}{x^{1/\alpha}(x+1)}$

We calculate this by integrating $\displaystyle\int_C dz\frac{1}{z^{1/\alpha}(z+1)}$ along the following contour



the branch cut is along the negative imaginary axis.

Thus we have (P stands for principle value)

$$\int_{-\infty}^{0} dx\frac{\mathrm{P}}{e^{\pi i/\alpha}\left(-x\right)^{1/\alpha}(x+1)}+\int_0^\infty dx\frac{1}{x^{1/\alpha}(x+1)}+\int_{|z|=R} dz\frac{1}{z^{1/\alpha}(z+1)}+\int_{|z|=\varepsilon} dz\frac{1}{z^{1/\alpha}(z+1)}$$
$$=\pi i\mathrm{Res}\left[\frac{1}{z^{1/\alpha}(z+1)},z=-1\right]$$

$$e^{-\pi i/\alpha}\int_0^\infty dx\frac{\mathrm{P}}{x^{1/\alpha}(1-x)}+\int_0^\infty dx\frac{1}{x^{1/\alpha}(1+x)}=\pi ie^{-\pi i/\alpha}$$

$$\int_0^\infty dx\frac{\mathrm{P}}{x^{1/\alpha}(1-x)}+e^{\pi i/\alpha}\int_0^\infty dx\frac{1}{x^{1/\alpha}(1+x)}=\pi i$$

$$\int_0^\infty dx\frac{\mathrm{P}}{x^{1/\alpha}(1-x)}+\left[\cos\left(\frac{\pi}{\alpha}\right)+i\sin\left(\frac{\pi}{\alpha}\right)\right]\int_0^\infty dx\frac{1}{x^{1/\alpha}(1+x)}=\pi i$$

Now I will equate real and imaginary parts of both sides. But first, let me call

$$I=\int_0^\infty dx\frac{1}{x^{1/\alpha}(1+x)} \qquad J=\int_0^\infty dx\frac{\mathrm{P}}{x^{1/\alpha}(1-x)}$$

$$\int_0^\infty dx\frac{\mathrm{P}}{x^{1/\alpha}(1-x)}+\left[\cos\left(\frac{\pi}{\alpha}\right)+i\sin\left(\frac{\pi}{\alpha}\right)\right]\int_0^\infty dx\frac{1}{x^{1/\alpha}(1+x)}=\pi i$$

$$J+I\left[\cos\left(\frac{\pi}{\alpha}\right)+i\sin\left(\frac{\pi}{\alpha}\right)\right]=\pi i$$

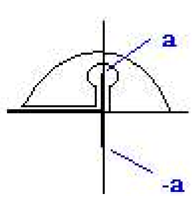
$$J+I\cos\left(\frac{\pi}{\alpha}\right)=0,\quad I\sin\left(\frac{\pi}{\alpha}\right)=\pi i$$

$$I=\frac{\pi}{\sin\left(\frac{\pi}{\alpha}\right)},\quad J=-\pi\cot\left(\frac{\pi}{\alpha}\right)$$

L. $$\int_0^\infty dx\ln\left(\frac{x^2+a^2}{x^2}\right)\cos(vx)$$

First lets consider the ancillary integral $$\int_C dz\ln\left(\frac{(z+ia)(z-ia)}{z^2}\right)e^{ivz}$$

using the following contour



Let me call C1 the interier circular integral, and C2 the exterior one. In order to ensure that positive values of x give real values of the ln, I think I will define the angles from a to run from $-\pi$ to π in a CCW fashion, and the angles from (-a) to run from $-\pi$ to π in CCW fashion. The angle from 0 will run from $-\pi$ to π in a CCW direction. That way, the angle at all points on the real axis will add to 0, giving a completely real ln( ). There seems to be some problem when I define angles to decrease in the CCW direction.

Now we have

$$\int_{-\infty}^{0} dx\ln\left(\frac{(x+ia)(x-ia)}{x^2}\right)e^{ivx} + i\int_0^a dy\ln\left(\frac{(a+y)e^{\pi i}(a-y)e^{\pi i}}{y^2 e^{\pi i}}\right)e^{-vy} + \int_{C_1} dz\ln\left(\frac{(z-ia)(z+ia)}{z^2}\right)e^{ivz}$$
$$+i\int_a^0 dy\ln\left(\frac{(a+y)e^{-\pi i}(a-y)e^{\pi i}}{y^2 e^{\pi i}}\right)e^{-vy} + \int_0^{\infty} dx\ln\left(\frac{(x+ia)(x-ia)}{x^2}\right)e^{ivx} + \int_{|z|=R} dz\ln\left(\frac{(z+ia)(z-ia)}{z^2}\right)e^{ivz} = 0$$

$$\int_{-\infty}^{0} dx\ln\left(\frac{(x+ia)(x-ia)}{x^2}\right)e^{ivx} + i\int_0^a dy\left\{\ln\left(\frac{(a+y)(a-y)}{y^2}\right)+\pi i\right\}e^{-vy}$$
$$+i\int_a^0 dy\left\{\ln\left(\frac{(a+y)(a-y)}{y^2}\right)-\pi i\right\}e^{-vy} + \int_0^{\infty} dx\ln\left(\frac{(x+ia)(x-ia)}{x^2}\right)e^{ivx} = 0$$

$$\int_{-\infty}^{\infty} dx\ln\left(\frac{(x+ia)(x-ia)}{x^2}\right)e^{ivx} + i\int_0^a dy\{\pi i\}e^{-vy} - i\int_0^a dy\{-\pi i\}e^{-vy} = 0$$

$$2\int_0^{\infty} dx\ln\left(\frac{a^2+x^2}{x^2}\right)e^{ivx} - 2\pi\int_0^a dye^{-vy} = 0$$

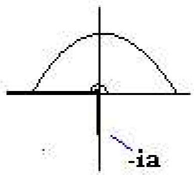
$$\int_0^{\infty} dx\ln\left(\frac{a^2+x^2}{x^2}\right)e^{ivx} = -\pi\int_0^a dye^{-vy} = \frac{\pi}{-v}\left[e^{-av}-1\right] = \frac{\pi}{v}\left[1-e^{-av}\right]$$

M. $\int_0^{\infty} dx\tan^{-1}(a/x)\sin(vx)$

Recalling $\tan^{-1}(z) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)$

We can write this as $\int_0^{\infty} dx\tan^{-1}(a/x)\sin(vx) = \frac{i}{2}\int_0^{\infty} dx\ln\left(\frac{i+a/x}{i-a/x}\right)\sin(vx) = \frac{i}{2}\int_0^{\infty} dx\ln\left(\frac{x-ia}{x+ia}\right)\sin(vx)$

We can calculate this one by considering the auxilliary integral $\int_C dz\ln\left(\frac{z+ia}{z}\right)e^{ivz}$

along the contour  Thus we have

$$\int_{-\infty}^{0} dx \ln\left(\frac{x+ia}{x}\right)e^{ivx} + \int_{|z|=\varepsilon} dz \ln\left(\frac{z+ia}{z}\right)e^{ivz} + \int_{0}^{\infty} dx \ln\left(\frac{x+ia}{x}\right)e^{ivx} + \int_{|z|=R} dz \ln\left(\frac{z+ia}{z}\right)e^{ivz} = 0$$

$$\int_{0}^{\infty} dx \ln\left(\frac{-x+ia}{-x}\right)e^{-ivx} + \int_{0}^{\infty} dx \ln\left(\frac{x+ia}{x}\right)e^{ivx} = 0$$

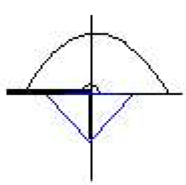
$$\int_{0}^{\infty} dx \ln\left(\frac{-x+ia}{-x}\right)\left[\cos(vx) - i\sin(vx) + \right] + \int_{0}^{\infty} dx \ln\left(\frac{x+ia}{x}\right)\left[\cos(vx) + i\sin(vx)\right] = 0$$

$$\int_{0}^{\infty} dx \ln\left(\frac{x+a^2}{x^2}\right)\cos(vx) - i\int_{0}^{\infty} dx \ln\left(\frac{x+ia}{x-ia}\right)\sin(vx) = 0$$

$$\frac{i}{2}\int_{0}^{\infty} dx \ln\left(\frac{x+ia}{x-ia}\right)\sin(vx) = \frac{1}{2}\int_{0}^{\infty} dx \ln\left(\frac{x+a^2}{x^2}\right)\cos(vx)$$

$$\int_{0}^{\infty} dx \tan^{-1}(x/a)\sin(vx) = \frac{\pi}{2v}\left[1 - e^{-va}\right]$$

Here we made an assumption that we could use the typical identities addition and subtraction ln( ) identities



The blue lines represent (from the bottom and going CCW) x+ia, x, -x, -x+ia. Now let us verify the identity that we used above.

$$\ln\left(\frac{x+ia}{x}\right) + \ln\left(\frac{-x+ia}{-x}\right)$$

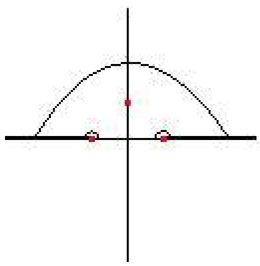
$$\ln\left(\sqrt{x^2+a^2}\,e^{i\tan^{-1}(1/x)}\frac{1}{|x|}\right) + \ln\left(\sqrt{x^2+a^2}\,e^{i\left[\pi-\tan^{-1}(1/x)\right]}\frac{1}{|x|}e^{-\pi i}\right)$$

$$\ln\left(\frac{\sqrt{x^2+a^2}}{|x|}\right) + i\tan^{-1}(1/x) + 1\left(\frac{\sqrt{x^2+a^2}}{|x|}\right) + i\left[\pi - \tan^{-1}(1/x) - \pi\right]$$

$$2\ln\left(\frac{\sqrt{x^2+a^2}}{|x|}\right) = \ln\left(\frac{x^2+a^2}{x^2}\right)$$

So there we go!

N. $\int_0^\infty dx \frac{\ln\left|x^2-a^2\right|}{x^2+b^2}$



Again, we'll represent x - a, and x + a as directed line segments from a, and -a respectively. Thus, in order for each point in the plane to correspond to a single complex number we have to make the following branch cuts - note that we can't join the two points by a cut because you could still make loops around the cut that would give you different values of a specific complex number. We require that ln(x^2-a^2) = ln[(x-a)(x+a)] be real on the real axis for x > a so we can measure both angles from 0 on the right of each branch point. The integrals around the branch points will be equal to 0 as usual. Finally, we'll look at the integral without the absolute value signs.

$$\int_0^\infty dx \frac{\ln(x^2-a^2)}{x^2+b^2} = \frac{1}{2}\int_{-\infty}^\infty dx \frac{\ln\left[(x+a)(x-a)\right]}{x^2+b^2}$$

$$\frac{1}{2}\int_{-\infty}^{-a} dx \frac{\ln\left[|x+a|e^{\pi i}|x-a|e^{\pi i}\right]}{x^2+b^2} + \frac{1}{2}\int_{-a}^{a} dx \frac{\ln\left[|x+a||x-a|e^{\pi i}\right]}{x^2+b^2} + \frac{1}{2}\int_{a}^{\infty} dx \frac{\ln\left[|x+a||x-a|\right]}{x^2+b^2} = 2\pi i \mathrm{Res}\left[\frac{1}{2}\frac{\ln\left[(x+a)(x-a)\right]}{x^2+b^2}, x=bi\right]$$

$$\frac{1}{2}\int_{-\infty}^{-a} dx \frac{\ln\left|x^2-a^2\right|+2\pi i}{x^2+b^2} + \frac{1}{2}\int_{-a}^{a} dx \frac{\ln\left|x^2-a^2\right|+\pi i}{x^2+b^2} + \frac{1}{2}\int_{a}^{\infty} dx \frac{\ln\left|x^2-a^2\right|}{x^2+b^2} = \pi i \mathrm{Res}\left[\frac{\ln\left[(x+a)(x-a)\right]}{(x+ib)(x-ib)}, x=bi\right]$$

$$\frac{1}{2}\int_{-\infty}^{\infty} dx \frac{\ln\left|x^2-a^2\right|}{x^2+b^2} + \frac{1}{2}\int_{-\infty}^{-a} dx \frac{2\pi i}{x^2+b^2} + \frac{1}{2}\int_{-a}^{a} dx \frac{\pi i}{x^2+b^2} = \pi i \frac{\ln\left[\sqrt{a^2+b^2}e^{i\tan^{-1}(b/a)}\sqrt{a^2+b^2}e^{i(\pi-\tan^{-1}(b/a))}\right]}{2ib}$$

$$\int_0^\infty dx \frac{\ln\left|x^2-a^2\right|}{x^2+b^2} + \pi i \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\Bigg|_{-\infty}^{-a} + \frac{\pi i}{2}\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\Bigg|_{-a}^{a} = \frac{\pi}{2b}\ln\left[\sqrt{a^2+b^2}\sqrt{a^2+b^2}e^{i\pi}\right]$$

$$\int_0^\infty dx \frac{\ln\left|x^2-a^2\right|}{x^2+b^2} + \pi i \frac{1}{b}\left\{-\tan^{-1}\left(\frac{a}{b}\right)+\frac{\pi}{2}\right\} + \frac{\pi i}{2}\frac{1}{b}2\tan^{-1}\left(\frac{a}{b}\right) = \frac{\pi}{2b}\left\{\ln\left[a^2+b^2\right]+\pi i\right\}$$

$$\int_0^\infty dx \frac{\ln\left|x^2-a^2\right|}{x^2+b^2} + \frac{\pi^2 i}{2b} = \frac{\pi}{2b}\ln\left[a^2+b^2\right] + \frac{\pi^2 i}{2b}$$

$$\int_0^\infty dx \frac{\ln\left|x^2-a^2\right|}{x^2+b^2} = \frac{\pi}{2b}\ln\left[a^2+b^2\right]$$

o. $$\int_0^\infty dx \frac{\ln(a^2+x^2)}{x^2+1} = \pi \ln(1+|a|)$$

This is easily established by differentiating with respect to a, evaluating the new integral and then integrating that result with respect to a. We can determine the constant of integration by letting a = 1. The result we have already calculated πln(2)

In general, any contour can be used, you just have to go around the singularities, branch cuts; you want to generally make it a closed loop so that you can equate it to the sum of the residues on the inside or outside, and you want the parts of the contour that don't end up being equal to the integral, or integral x phase factor, etc., to be easily evaluatable, i.e. they go to zero, or they're just easy to calculate. So you can have circles, semi-circles, arcs, rectangles, whatever.

H. Using Residues to Evaluate Sums

First let's note a few facts:

If f(z) has a simple pole at z = z0 and if it its residue is R, then f(az) has a simple pole at z0/a and a residue of R/a.

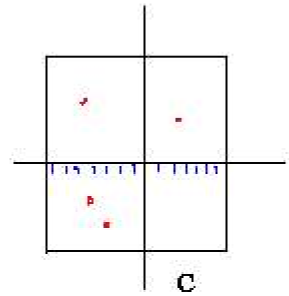
$n_F(z) = \dfrac{1}{e^{\beta z}+1}$ has residues at the values $\beta z = (2n+1)\pi i \to z = (2n+1)\pi i/\beta = i\omega_n$ and a residue of $1/\beta$

$n_B(z) = \dfrac{1}{e^{\beta z}-1}$ has residues at the values $\beta z = 2n\pi i \to z = 2n\pi i/\beta = i\nu_n$ and a residue of $1/\beta$

$\cot(z)$ has residues at the values $z = n\pi$ and a residue of 1

$\csc(z)$ has residues at the values $z = n\pi$ and a residue of $(-1)^n$

Consider integral $\int_C \pi \cot(\pi\cdot z)\cdot f(z)\,dz$ where C is the contour shown; the blue marks are the poles of cot(pi*z) - the integers, and the red marks are the poles of f(z), presumed not to be integers, though they could lie on the real axis otherwise.



For certain f(z)'s - certainly for rational f's whose denominator exeeds its numerator by a power greater than 1 - if you extend the contour out to infinity, you'll see that it goes to zero, since |coth(z)| < 1 on the boundary, and so you have that

$$\sum_{\text{all}} \mathrm{Res}[(\pi)\cdot\cot(\pi\cdot z)\cdot f(z)] := \sum_{\text{all}} \mathrm{Res}\left(\pi\cdot\frac{\cos(\pi\cdot z)}{\sin(\pi\cdot z)}\cdot f(z)\right) \quad \text{equals} \quad \sum_{n=-\infty}^{\infty} f(n) + \sum_{\text{fpoles}} \mathrm{Res}(\pi\cdot\cot(\pi\cdot z)\cdot f(z))$$

The last sum is over the poles of f(z)

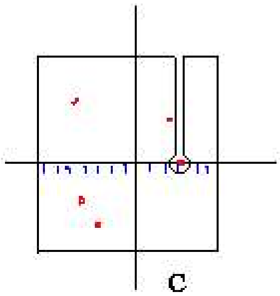
to evaluate the residue of the function at the integer points, you just take the derivative of the denominator and evaluate at the pole (remember), and you get just f(n) - that's only if f(z) doesn't have a pole there too - because taking the derivative won't get you the residue if the pole isn't simple.

And this equals 0, so ...

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{\text{f poles}} \mathrm{Res}\left[-\pi \coth(\pi z) f(z)\right]$$

A similar analysis will show that $$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = \sum_{\text{f poles}} \mathrm{Res}\left[-\pi \,\mathrm{csch}(\pi z) f(z)\right]$$

To get around the problems of f(z) having poles at integers, most often z = 0, try this contour.



As is familiar, we can use the top two distribution functions to find the sum of functions defined over complex frequencies. And now let's consider evaluating inverse FT sums.

Consider the FT pair $F(n) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{-in\varphi} f(\varphi)$ $\qquad f(\varphi) = \sum_{n=-\infty}^{\infty} e^{in\varphi} F(n)$

Given F(n) we can sometimes determine the explicit form for f(ϕ). Consider the integral below, where C is the usual box contour.

$$\frac{1}{2i}\int_C dz \frac{e^{-i\pi z}}{\sin(\pi z)} e^{iz\varphi} F(z)$$

We note that the integrand has poles at πn, thanks to the sin(z) in the denominator, and wherever f(z) has poles. The residue at at πn is

$$\frac{1}{2i}\frac{e^{-i\pi n} e^{in\varphi} F(n)}{(-1)^n \pi} = \frac{1}{2\pi i} e^{in\varphi} F(n)$$

And so the integral over the contour will come out to be the sum below. But note that its also zero, and this is proved in the book by actually integrating along the contour itself.

$$2\pi i\left\{\sum_{n=-\infty}^{\infty} \frac{1}{2\pi i} e^{in\varphi} F(n) + \mathrm{Res}\left(\frac{1}{2i}\frac{e^{-i\pi z} e^{iz\varphi} F(z)}{\sin(\pi z)}, \text{poles of } F\right)\right\} = 0$$

$$\sum_{n=-\infty}^{\infty} e^{in\varphi} F(n) + \mathrm{Res}\left(\frac{\pi e^{-i\pi z} e^{iz\varphi} F(z)}{\sin(\pi z)}, \text{poles of } F\right) = 0$$

And so we have finally that,

$$f(\varphi)=\sum_{n=-\infty}^{\infty} e^{in\varphi}F(n)=-\mathrm{Res}\left(\frac{\pi e^{-i\pi z}e^{iz\varphi}F(z)}{\sin(\pi z)},\text{poles of }F\right)$$

Note that we have to use this particular integrand because otherwise, the integrand wouldn't drop to 0 along the contours of the box.

Note that this is the same as the Matsubara technique I think.

I. Weierstrauss' Theorem for Infinite Products

Let f(z) be analytic for all z, and suppose that it has simple zeros at a1, a2, ... Then it can be expressed as an infinite product of the form

$$f(z)=f(0)e^{f'(0)z/f(0)}\prod_{k=1}^{\infty}\left\{\left(1-\frac{z}{a_k}\right)e^{z/a_k}\right\}$$

An example of this is the function (apparently applying this theorem to sin(z)/z, and excluding the little region near z = 0)? In this case it does work out, and note that the product goes over ALL zeroes. This is apparent when you factor the productand. And note that labelling the zeroes 1, 2, ..., infinty - does't mean that there can't be zeros on both sides of the origin, just that we label them in an alternating fashion sort of.

$$\sin(z)=z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right)$$

$$\cosh(z)=\prod_{k=1}^{\infty}\left(1+\frac{4z^2}{(2k-1)^2\pi^2}\right)$$

Mittag Liefler's theorem

Suppose that the only singularities of f(z) in the finite z plane are the simple poles p_n with residues R_n. Then

$$f(z)=f(0)+\sum_n R_n\left\{\frac{1}{z-p_n}+\frac{1}{p_n}\right\}$$

As can be shown using Cauchy's theorem

J. Analytic Continuation

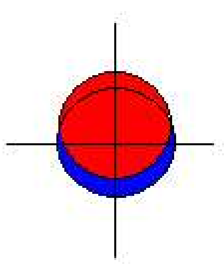
Let F1(z) be a function of z which is analytic in a region R1, and suppose that we can find a function F2(z) which is analytic in a region R2 (which overlaps with R1, or even that just their boundaries partly lie on top of eachother), and which is such that F1(z) = F2(z) in this region. Then we say that F2(z) is an analytic continuation of F1(z), which implies that there is a function F(z) analytic in the combined region R1 + R2 such that F(z) = F1(z) in R1 and F(z) = F2(z) in R2.

A function may be analytically continued into a new region in a successive, ring like manner. If such a function is analytically continued to a point along two different paths, the analytic continuation will be the same in both cases iff there is no singularity of the function enclosed by the two paths.

For example, consider that we knew the T. series expansion of a function about the point z = 0, which was analytic inside region B (the blue circle with radius 1). We would know that the series is valid inside this circle because using standard convergence tests, we'd be able to verify that the series converged and was differentiabe, etc., inside this circle. So we have,

$$f(z) = 1 + z + z^2 + \ldots$$

Now suppose that we wanted to know what the value of this function was at z = i. We obviously can't plug it into the present expansion, but we can analytically continue our expansion into the red region, R (circle with radius sqrt(5/4) centered at z = i/2). We would do this in the following manner.



Let $f_c(z) = a + b(z - i/2) + c(z - i/2)^2 + \ldots.$

be the analytically continued function expanded about z = i/2. We can determine the coefficients by matching up the function/derivative values at z = i/2.

$$f(i/2) = a, \quad f'(i/2) = b, \quad f''(i/2) = 2c \qquad \text{etc.}$$

Then having determined more or less, the coefficients, we can plug z = i into fc(z) to determine that the analytically continued function's value there.

Of course, no longer ignoring the obvious, we can state that inside B, the function is given by:

$f(z) = \dfrac{1}{1-z}$ And we could define this as the analytic continuation of our Taylor series function outside region B.

Note that its sometimes the case that a function can't be continued outside its original region of analyticity. In this case this region is called the natural boundary of the function. This would be the case for a function with singularities at every point on its boundary, or a branch cut everywhere, etc.

If F(z) is analytic inside a region R, and if it is equal to 0 along a strip enclosed in R, then F(z) = 0 everywhere in R. This can be straightforwardly proved using the ideas above. For, consider a point, z0 on the strip. We would T. expand our function there.

$$F_c(z) = b(z - z_0) + c(z - z_0)^2 + \ldots.$$

He says that we know the derivatives of Fc are all 0 at z0 as well. So that means that Fc(z) = 0 inside the region. Not sure why we know that exactly. But anyway, a consequence of this theorem is that if a function F1(z) analytic in R1 is equal to a function F2(z) analytic in R2 which overlaps, even if only on a strip, with R1, then F1(z) = F2(z) in R1 + R2.

$\cos^2(x)+\sin^2(x)=1$ along the real axis. Now let $F(z)=\cos^2(z)+\sin^2(z)-1$ since F(z) = 0 on the strip that is the real axis, and since F(z) is analytic everywhere, F(z) = 0 everwhere. So it is more generally true that

$$\cos^2(z)+\sin^2(z)=1$$

So generally speaking, if F1(z) is analytic in a region and equals F2(z) along some strip in that region, then F1(z) = F2(z). This naturally leads into Schwartz' reflection principle. It states that if a function, F(z) is analytic in a region of the u.h.p., and assumes real values on the x-axis, then its analytic continuation into the reflection of the region, R, about the x-axis is F*(z*). One may, I suppose, use this as a test for analyticity along the x-axis by forming the complex conjugate and taking the difference F(x+i0) - F*(x-i0). If the difference doesn't equal 0 then its not analytic, and we also see the discontinuity.

Consider the integral: $$F(t)=\int_{-\infty}^{\infty} dx\, e^{tx^2-x^4}$$

It is analytic for all t in the complex plane - in particular the real and imaginary axes. For imaginary t, it is difficult to find an asymptotic formula approximation but for real t it is easy, using the SP approximation. Nonetheless, while F(t) can be continued to the imaginary axis by replacing t with it, it doesn't seem that the same can be said of the asymptotic formula.

K. Fourier and Laplace Transforms

The fourier transform pair is:

$$F(\omega)=\int_{-\infty}^{\infty} dte^{-i\omega t}f(t) \qquad f(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}F(\omega)$$

Note that sometimes F(w) won't exist for a given f(t), and sometimes f(t) won't exist for a given F(w) according to the formula. But it is often the case that even if F(w) according to the formula, you can often find a function whose inverse FT equals f(t). Consider for example θ(t). The Fourier transform of θ(t) doesn't exist, yet there is a function: $i/(\omega+\iota\delta)$ whose inverse FT equals θ(t). Just keep it in mind. This sort of justifies playing loose with FT or inverse FT to find functions - they often exist, but I suppose must be analytically continued.

Consider a derivation of the inverse LT from the FT. Combining the two formulas above,

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \int_{-\infty}^{\infty} dt' e^{-i\omega t'} f(t')$$

Then let's suppose $f(t) = g(t)e^{-at}$ where g(t) is 0 for t < 0. Then we can write this as,

$$g(t)e^{-at} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \int_{0}^{\infty} dt' e^{-i\omega t'} g(t') e^{-at'}$$

$$g(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t + at} \int_{0}^{\infty} dt' e^{-i\omega t' - at'} g(t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{(a+i\omega)t} \int_{0}^{\infty} dt' e^{-(a+i\omega)t'} g(t')$$

Now we can change variables to s = a + iw.

$$g(t) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} e^{st} \int_{0}^{\infty} dt' e^{-st'} g(t') = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} e^{st} G(s)$$

And we have the desired inversion formula. Note that the role of 'a' is that FT of g(t)e^(-at) exists, which implies that a is large enough to damp the behavior of g(t). I wonder if it is possible to construct a guassian transform in a similar way.

$$g(t)e^{-at^2} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \int_{-\infty}^{\infty} dt' e^{-i\omega t'} g(t') e^{-at'^2}$$

$$g(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t + at^2} \int_{-\infty}^{\infty} dt' e^{-i\omega t' - at'^2} g(t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{at^2 + i\omega t} \int_{-\infty}^{\infty} dt' e^{-(at'^2 + i\omega t')} g(t')$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{a(t+i\omega/2a)^2 - \omega^2/4a} \int_{-\infty}^{\infty} dt' e^{-a(t+i\omega/2a)^2 + \omega^2/4a} g(t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{a(t+i\omega/2a)^2} \int_{-\infty}^{\infty} dt' e^{-a(t+i\omega/2a)^2} g(t')$$

Now let s = t + iw/2a; this implies that ds = idw/2a.

$$g(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{a(t+i\omega/2a)^2} \int_{-\infty}^{\infty} dt' e^{-a(t+i\omega/2a)^2} g(t')$$

Not working out I don't think.